2013

Zero Revolution

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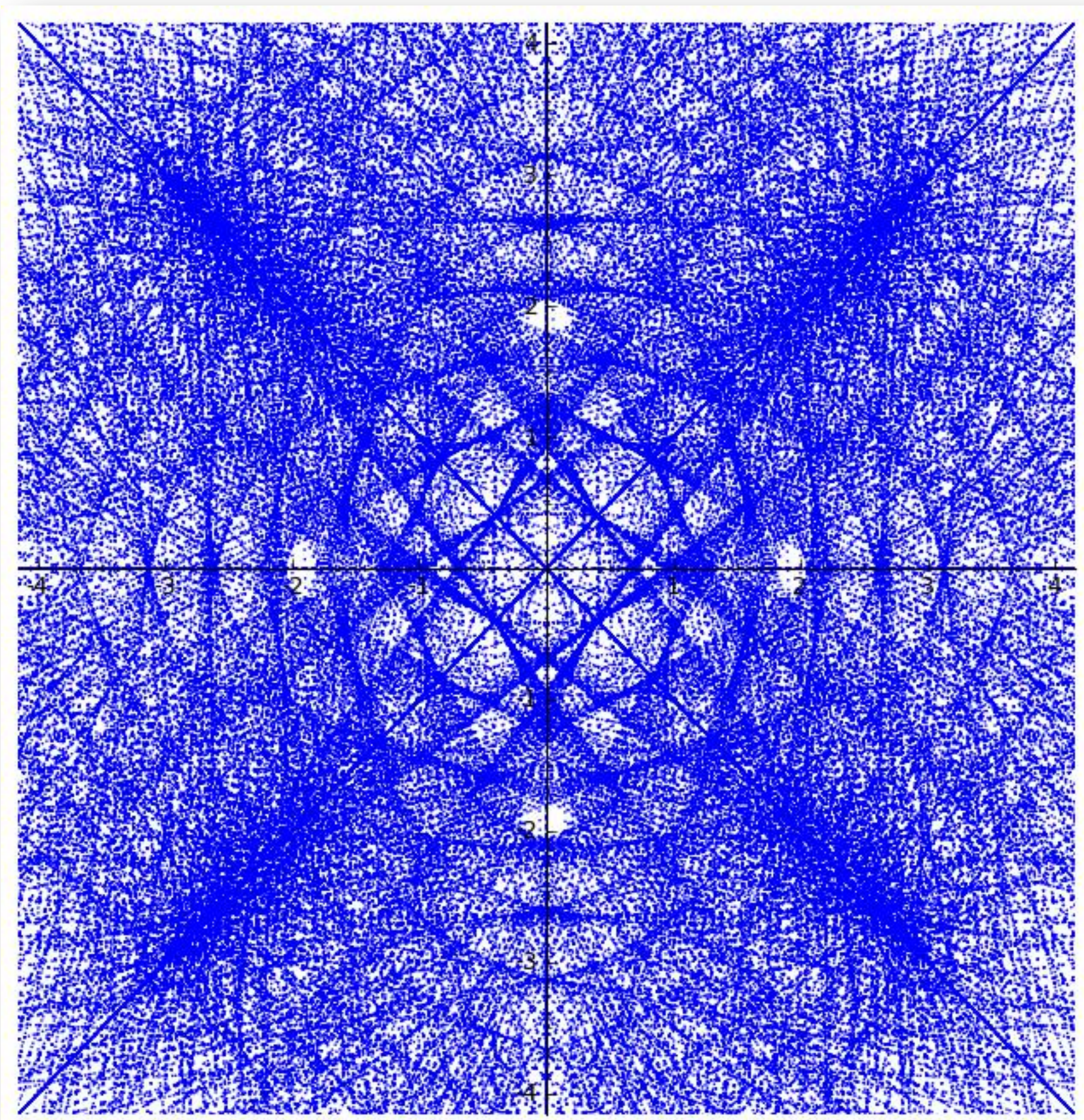
[ZERO REVOLUTION]

[A Study of Infinity, Zero, and Nature.]



Explaining the Universe Through Mathematics

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Chapter Zero: Zero Revolution

What is this Book?

Imagine you are a racecar driver. You must tune your engine for the big race. Unfortunately, you only have a little air in your tires. You may be able to start the race, and even make a few laps. But…eventually, time will take its toll and it will become impossible to win the race. Currently, math can describe many interesting characteristics of our world, but when it comes to certain operations it simply crashes and burns.

What is the state of math today?
Incomplete.

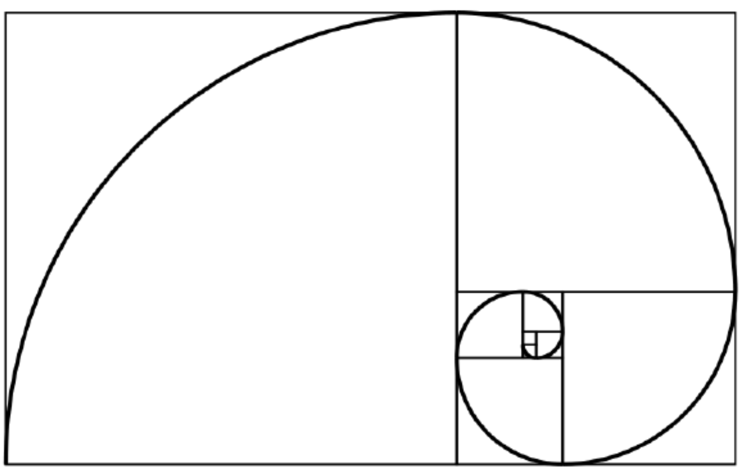
The purpose of this book is to take you, the reader, on a journey. You will see that math is not just numbers.

Math is everything.

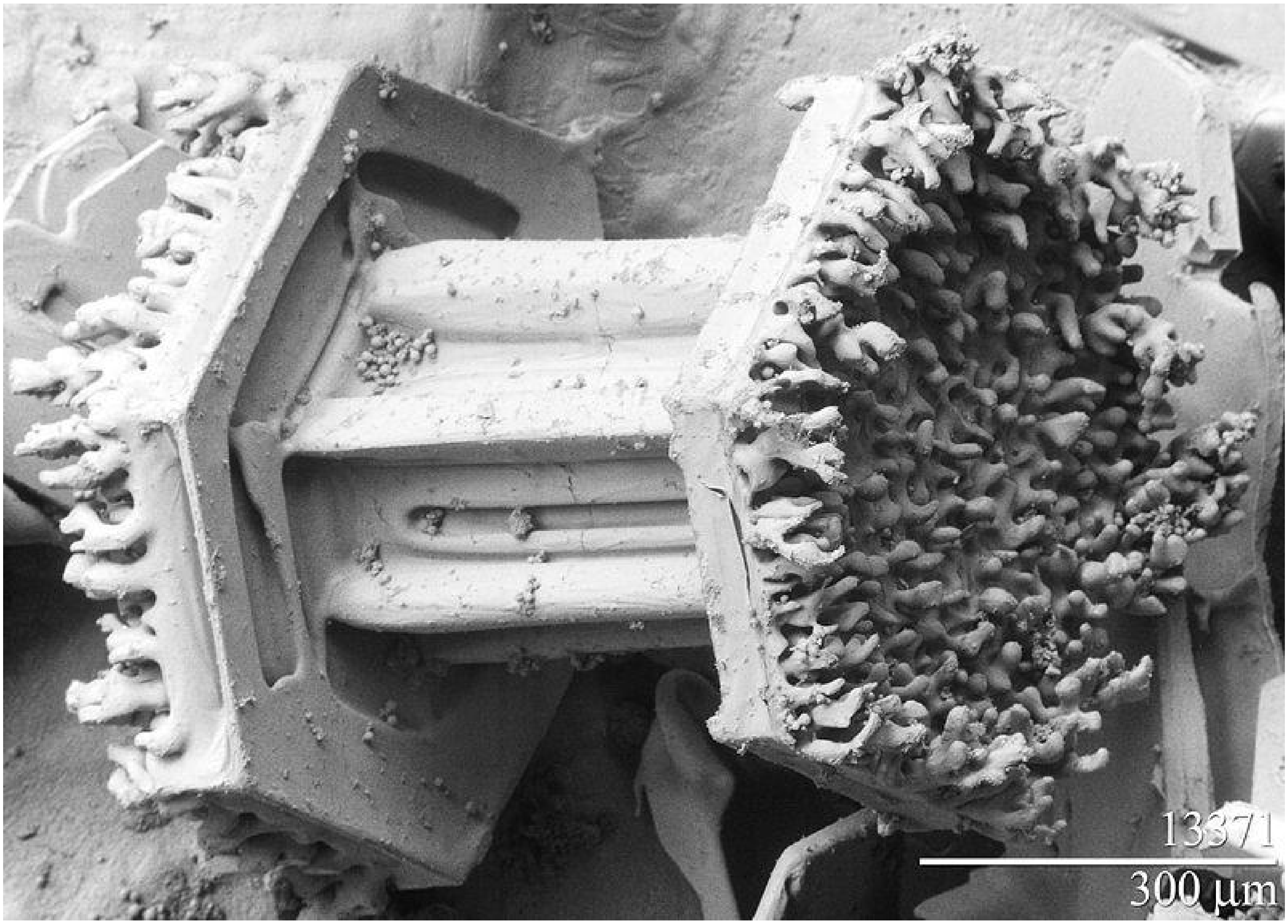
But, how can that be true, how can math be everything? It's not like there a math program running the universe? What am I saying?

Math can be found in nature over and over, at all levels. The same fundamental mathematical constants appear all over the world around us. From the behavior of an atom, to the structure of a nautilus shell, to the hurricane over Iceland (shown below), all the way to the formation of a galaxy, we can't hide from the same sightings.

Below is the *Golden Spiral*. It is a simple extension of the *Golden Ratio*, to which countless books have been dedicated. You can see the resemblance below to a storm over Iceland.

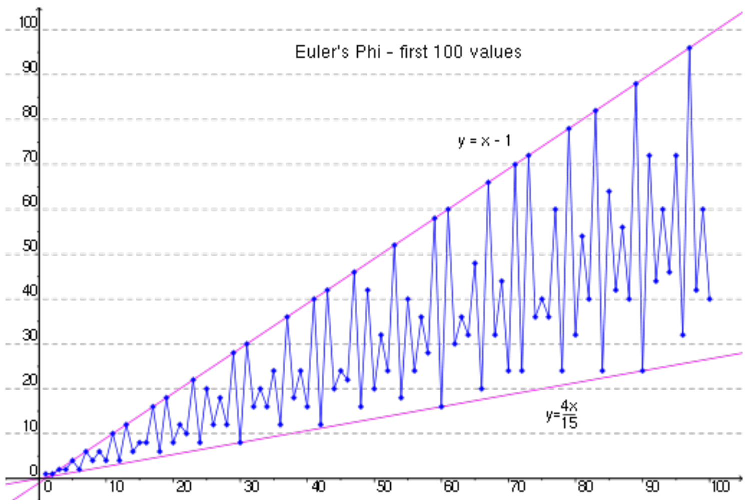


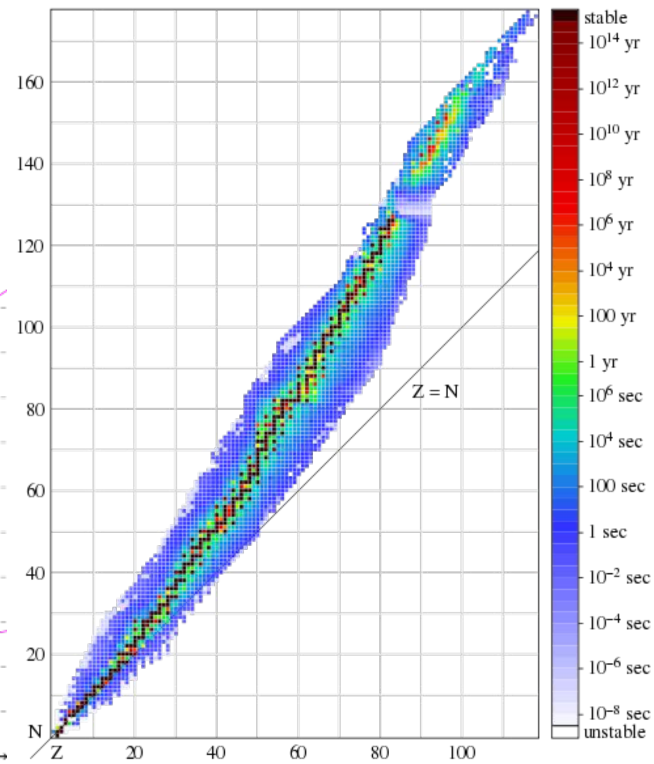
Have you ever looked closely at a snowflake? I mean really closely…



It looks like there is math within our entire world, both large and small. In fact, the “building blocks” of numbers are thought to be prime numbers, while the “building blocks” of matter is considered to be atoms. There have been numerous correlations between prime numbers and many aspects of particle physics, the study of atoms and sub-atomic particles.

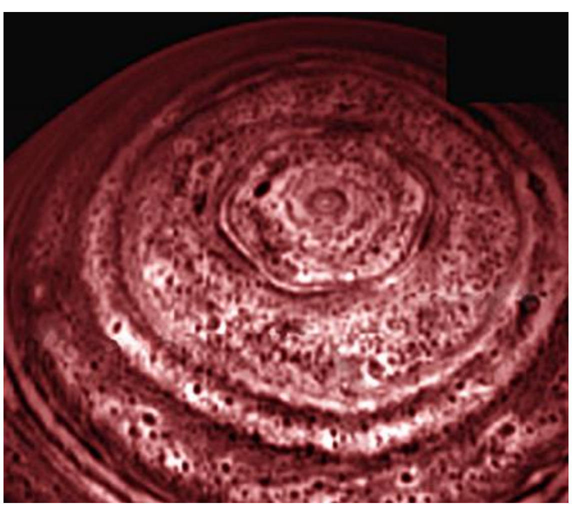
Below shows a graph based on pure mathematics (left) and a graph based on pure physics (right). The first graph, called the *Euler Phi Function*, is a straightforward mathematical function relating prime numbers and the second is the stability of every known atom.





Both of these graphs behave somewhat chaotically, but with some general pattern. There is no known formula for predicting exactly the future behavior of either of these graphs.

Our question is can we go beyond these interesting pictures and formalize some theory?



What is stopping us? Why we are still stuck with incomplete theories of our universe?
What if the very basic language we use did not have the adequate words for understanding our universe? Well, our language for describing the universe is not English, French, or Russian, but the common language of mathematics. What if our mathematics does not yet have all the necessary ideas to fully describe the world that our physicists study today? Or worse, what if at the very core of our mathematics, we have an incorrect concept or misunderstood idea?

I believe the later is the case, and the most misunderstood idea is:

ZERO

Logic

At the core of mathematics, there is logic. Our own sense of logic has helped us define the rules of mathematics. Therefore, if our logic is flawed in any way, we will propagate this mistake through our rules in mathematics.

Where do we get our sense of logic? We based our rules of logic on our own experiences. We believe that the logic that governs the universe is represented in mathematics. These rules governing the universe are the same logic we try to capture in mathematics. For example, if I give you an apple and you already have one apple, you have two apples. This logic is captured well in our current system of mathematics.

However, in the last century, we have noticed a lot of counter-intuitive behavior when scientists first began to study the macro and micro scale (large galaxies and tiny atomic behavior). For example, when we first discovered that supermassive black holes, it was very difficult to even comprehend how that could be possible.

Regardless of how strange we discover our universe to be, we must make sure that mathematics captures the logic of our **entire universe** at all scales. It is imperative that we use a system of mathematics that can be applied to accounting, physics, biology, and even **consciousness**; the fact that there exist self-aware beings.

Now, this seems like a tall order! But, there has already been a lot of progress in extending mathematics. Unfortunately, some concepts in mathematics are still taboo such as discussing the possibility of dividing by zero. Still, just recently, there have been a handful of mathematicians, like Jacob Czajko, that have come to the realization that we must explore previously forbidden operations in mathematics, such as division by zero, before we can fully unlock the mysteries of our universe.

I personally call the idea of redefining zero…

The Zero Revolution

Imaginary Numbers

The general idea behind an imaginary number is that it is the square root of -1. This is a very counter-intuitive idea because that would mean that a number multiplied by itself will yield a negative number. However, all the real numbers that we commonly encounter will equal a positive number when multiplied by itself. It seems there is no practical or real world application for this idea, and it can only be useful in theory, right?

This couldn't be further from the truth. Let's address the mysterious and exciting natural force called electricity and magnetism. Electrical engineering would be virtually impossible to describe without imaginary numbers. Imaginary numbers exist in nature and they are literally "imaginary". The real universe uses these imaginary numbers; perhaps we don't observe them directly in our daily live, but we feel their effects when we use electricity.

Indeed, the behavior of imaginary numbers exists in our universe and not only theoretically. It can be more eloquently stated:

Either imaginary numbers are "real" or our universe is imaginary.

Perhaps, redefining zero is necessary to further our understanding of physical behavior, just as the invention of imaginary numbers was necessary to describe electricity. What may first seem like a theoretical exercise may suddenly have practical applications.

Bernhard Riemann

Bernhard Riemann became famous for a myriad of work that could take an entire book to detail. One famous equation that he is responsible for is the Riemann Zeta Function. This function describes a set of very straightforward infinite series'. Interestingly, there are so many correlations to real world numbers from such a simple function.

Without giving the explicit formula, you can see how it is defined as I "plug in" in the parameter 1 and 2 below:

Using 1 as the Parameter:

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty;$$

Using 2 as the Parameter:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \approx 1.645;$$

The following table gives more results of the Riemann Zeta function and illustrates some of the fascinating correlations between the Riemann Zeta Function and the world of Physics.

Riemann Zeta Function Parameter	Value	Where it is seen in the real world
0	-1/2	Particle Spin
1	Infinity	Everywhere
3/2	2.612	Bose–Einstein condensates, spin-wave physics for magnetic systems
2	1.645	Solution to Basel Problem
3	1.202	Apery's Constant
4	1.0823	Stefan–Boltzmann law and Wien approximation in physics

We will later refer to this function in our talk about infinity. The main takeaway is that it would be a huge coincidence for such a basic function to find its way into so many physical laws around us if there wasn't a strong connection. As we continue, let's examine what other connections exist between nature and mathematics, and hopefully figure out a possible way to put all these pieces together.

The idea that the entire universe is a mathematical structure is not a new. In fact, theoretical physicist Max Tegmark proposed the mathematical universe hypothesis (**MUH**). This states that o*ur external physical reality is a mathematical structure*. That is, the physical universe *is mathematics* in a well-defined sense, and "in those [worlds] complex enough to contain self-aware substructures [they] will subjectively perceive themselves as existing in a physically 'real' world". This book takes a baby-step by suggesting that there exists a mathematical substructure that must be self-aware to support this type of theory. This substructure is a new definition of the concept of *Zero*.

Chapter 1: The Problem with Zero

Arithmetic

Let’s look how we solve some simple equations. Let’s look at the example:

Does 7 = 6?

We know the answer is **NO**, but let’s try some of the legal mathematical operations just for fun! Let’s multiply by 2 on both sides.

2 X 7 = 2 X 6 ?
14 = 12 ?
STILL NOT TRUE

We see that multiplying by 2 doesn’t help. However, what if we multiply by zero? What happens? Let’s try!

0 X 7 = 0 X 6 ?
0 = 0
TRUE?

Oh dear, we definitely can’t allow anyone to multiply by zero on both sides of the equation. That breaks everything we worked on. Let’s just write a rule that says, “You can multiply by any number except Zero”. Good job everyone; let’s get a beer!

Perhaps, you’ve caught the subtle sarcasm above. Now, it’s time to look at a few more examples.

Dividing by Zero

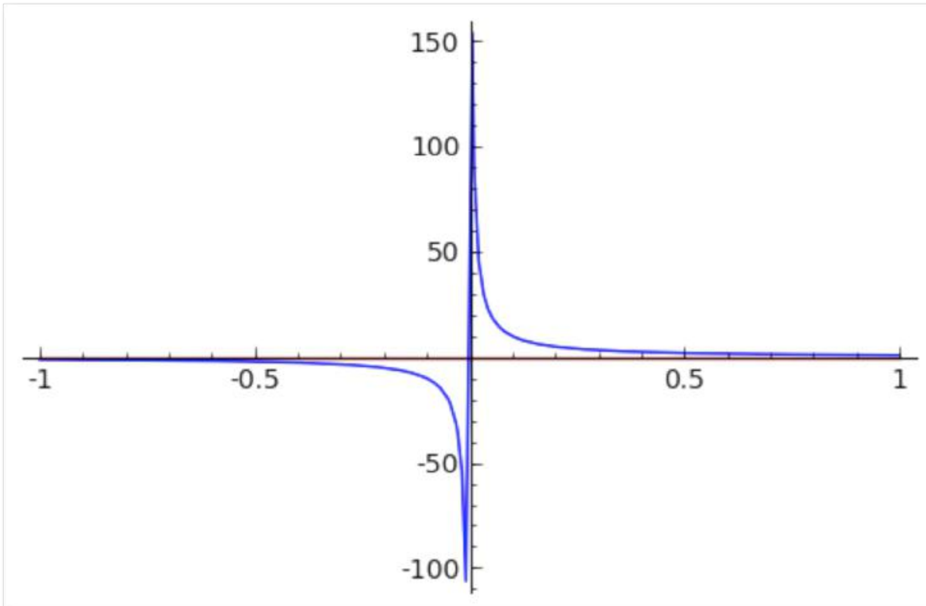
Zero was invented a long time ago, a very long time ago.

There is a strong consensus on how zero operates. That consensus is that if you multiply by zero, you get zero. There is one caveat; just don’t divide by it.

So, what is 1/0? According to mathematics it is **undefined**. Why is it undefined?

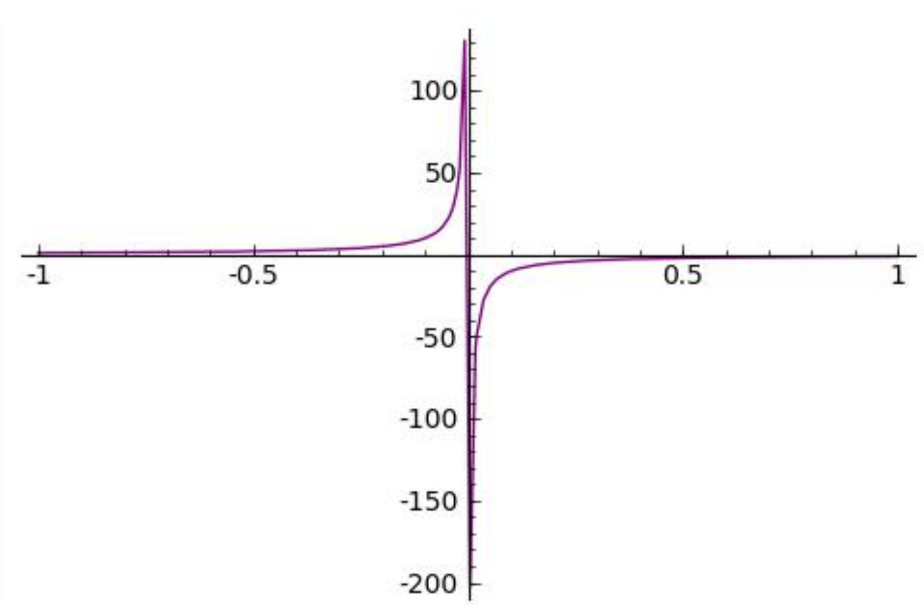
Well, let's examine the function and to see its behavior. As we get closer to 0 we see that the results diverge from one another. When approaching zero from one direction we get positive infinity and the other direction we get negative infinity...

1 Divided by X (real numbers)



How about when we include the complex numbers (both real number and imaginary numbers)?

1 Divided by X (imaginary numbers)



The axis' are reversed because 1/i =-i. Still, depending on whether we approach zero from the positive or negative direction, we arrive at a different solution.

Lack of Symmetry

Current mathematics still has many logical holes in it. For example:

0 X 0 = 0
but…

$$\sqrt{0} = ?$$

Answer: **Undefined** (you would think zero)

Why do mathematicians need to resort to the "undefined card"? Take a look at what happens to a few iterations of the square root of x. Our next chapter will examine, in detail, the behavior of the square root operation.

The problem with Infinity

In fact, our mathematics is so weak when it comes to addressing issues like zero and infinity we do not have a *defined* solution for the following:

$$1^{\infty} = \text{undefined}$$

Speaking candidly, if we as a society plan to make progress, we must do significantly better than *undefined*. Many of the most perplexing questions are still left unsolved, or worse, abandoned.

One motivation to writing this book is to inspire truly creative thinking. Rather than memorizing a set of existing rules, I hope to inspire a new breed of mathematicians that will focus on the "core" of mathematics and build a stronger foundation from which we can all benefit. This short book is a call to creative thinkers who may expand this concept into a formal theory.

Chapter 2: Infinite Roots

Introduction

Before we begin to experiment with any concept, it is important to identify our goals. In this chapter, our aim will be to study the behavior of the square root. How does it affect real numbers? How does it affect imaginary numbers? How does it behave when we approach Zero?

Let's try to identify patterns that emerge. More interestingly, can we connect any of these graphs with famous mathematical constants? Perhaps, we will even notice some parallels to the physical world.

A Humble Beginning…

I originally wanted to have a separate chapter on defining zero and infinity. I then realized this was impossible and they should be described in parallel. It would be analogous to describing positive numbers separately from negative numbers. It is possible, but ill-advised. Our very first experiment will examine Zero after doing an infinite set of square root operations.

This first exercise will be to look at both how square roots affect real numbers (blue curve) and imaginary numbers (red line).

Iterations of the Square Root of X

$\sqrt{x}, \sqrt{\sqrt{x}}, \sqrt{\sqrt{\sqrt{x}}}, \ldots$

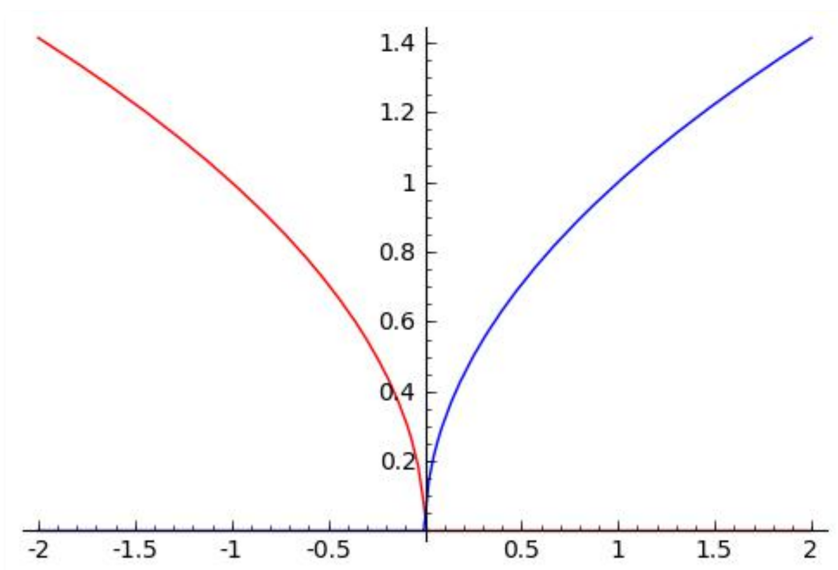


Figure 1: After One Iteration

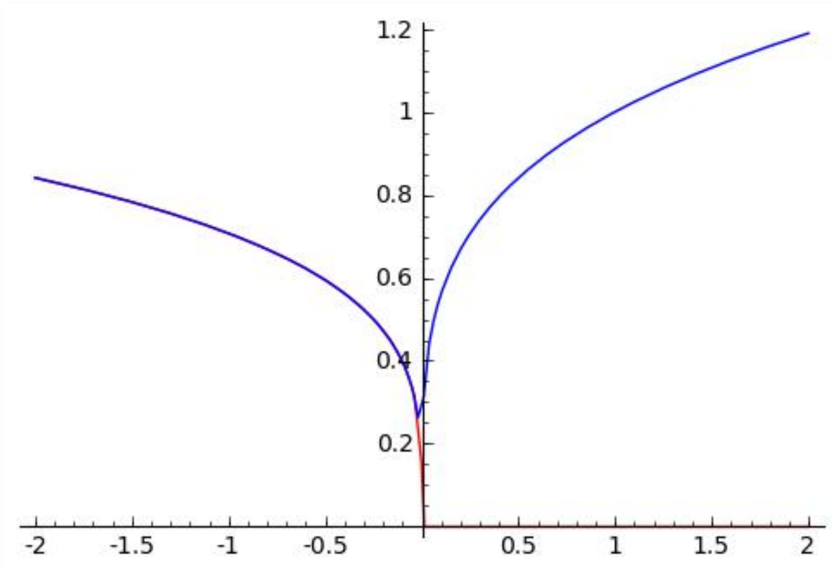


Figure 2: After Two Iterations

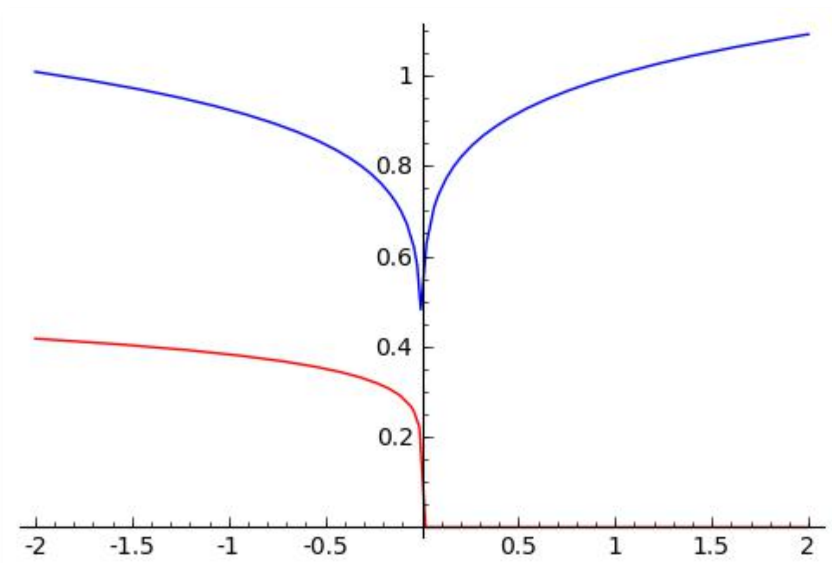


Figure 3: After Three Iterations

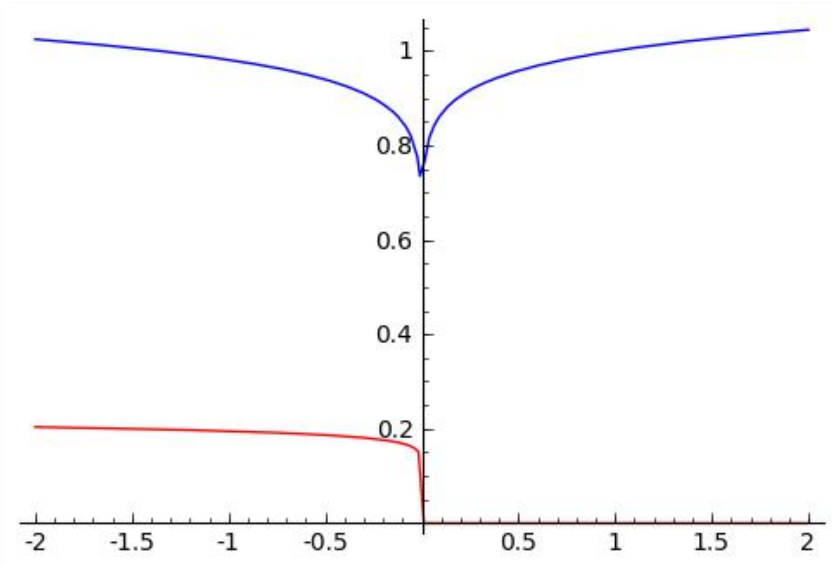


Figure 4: After Four Iterations

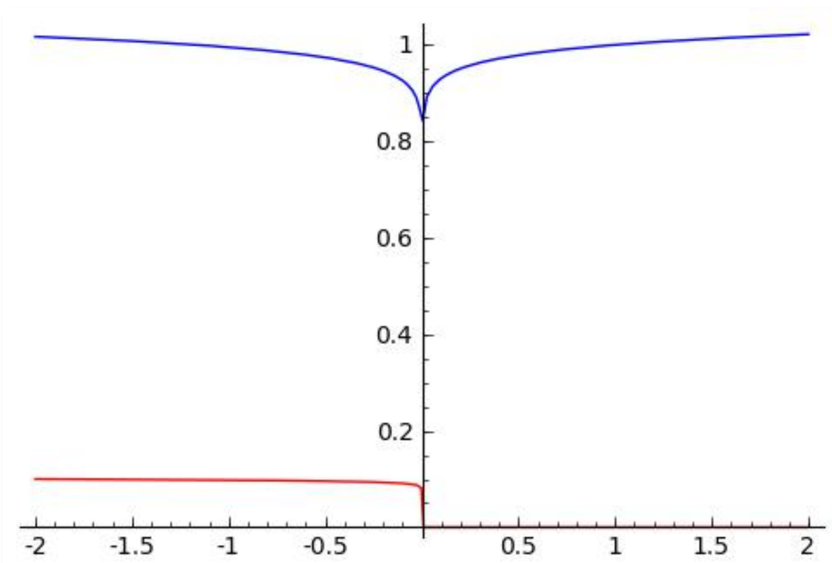


Figure 5: After Five Iterations

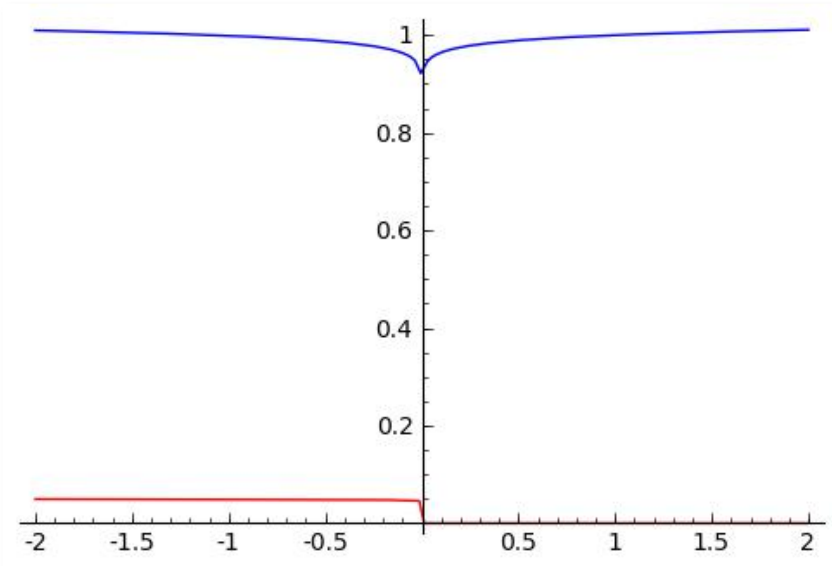


Figure 6:After Six Iterations

So really we could say:

$$\lim_{n\to\infty} \sqrt[n]{0} = 1$$

Interestingly we do say this, but in a more common form:

$$\mathbf{0^0 = 1}$$

First, this result is shocking to some. How do we end up with one by taking the "zeroeth" power of zero?

Still, we haven't gotten to the heart of the matter. What is really going on at zero? Can we just say a single square root is undefined but an infinite amount is defined, very definitely at 1?

This brings us to our first question:

How does the $\sqrt{0}$ behave?

Furthermore, is it possible to answer additional questions such as…

$$\lim_{n\to\infty/2} \sqrt[n]{0} = ?$$

Can we create a map to describe how zero and infinity behave together? Let's begin the adventure...

Enter the Principal Root

Did we miss anything? Zero is neither positive, nor negative. Why would zero to the zero power equal a positive number?

Consider the follow:

$$\sqrt{9}=3$$

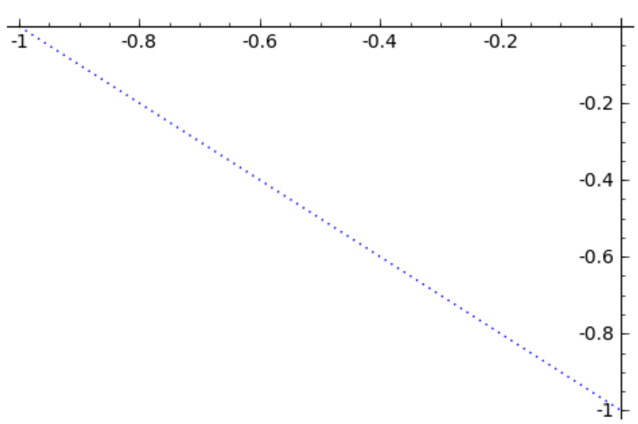
But also, $\sqrt{9}=-3$

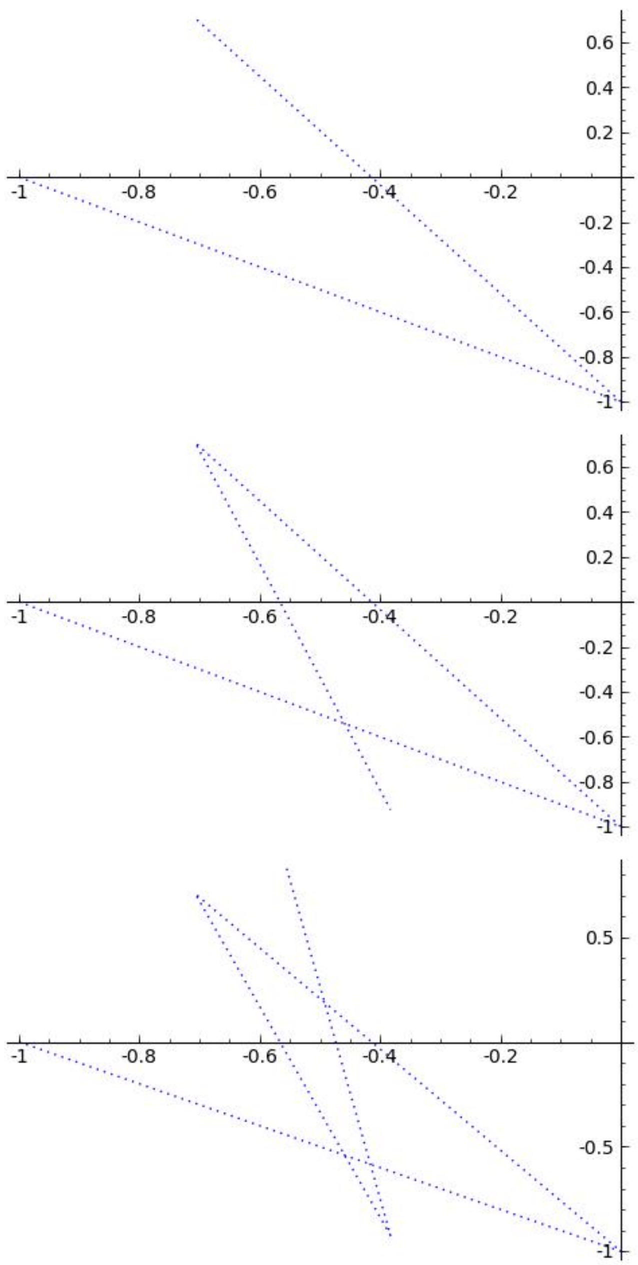
Formally, it is written as: $\sqrt{9}=\pm 3$

Each time we compute a square root there are **two roots**. In general, we call the positive value the *principal root*. So let's examine the behavior if we **don't** take the principal root and instead take the other path.

Iterations of the Square Root of X (NOT using the principal root)

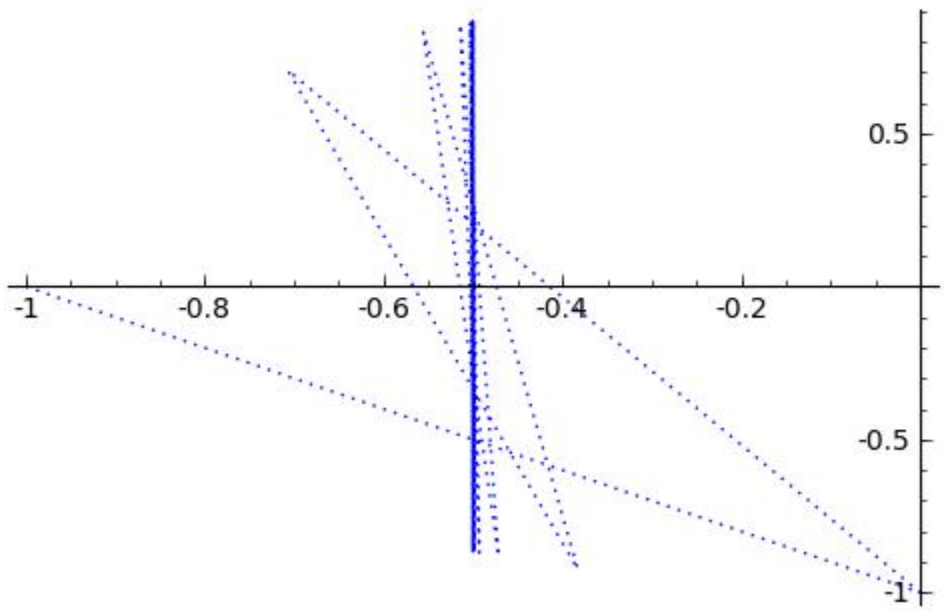
$\sqrt{1},\sqrt{\sqrt{1}},\sqrt{\sqrt{\sqrt{1}}},...$ becomes $-\sqrt{1},-\sqrt{-\sqrt{1}},-\sqrt{-\sqrt{-\sqrt{1}}},...$





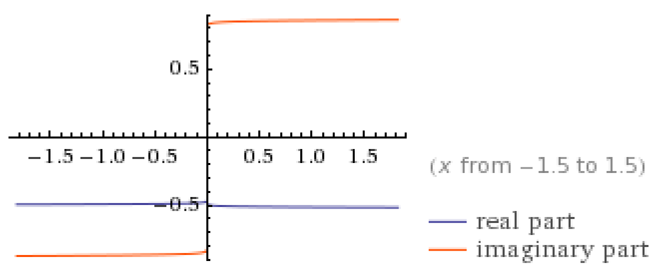
After about 20 iterations, we bounce between two points:

$$-.5 \pm \frac{\sqrt{3}}{2} i$$



A graph, generated from wolframalpha.com, shows the same behavior in another format:

$$-\sqrt{-\sqrt{-\sqrt{-\sqrt{-\sqrt{-\sqrt{-\sqrt{x}}}}}}}$$



In fact, no matter what number we start with we will always end up oscillating between the same two values. What are these values?

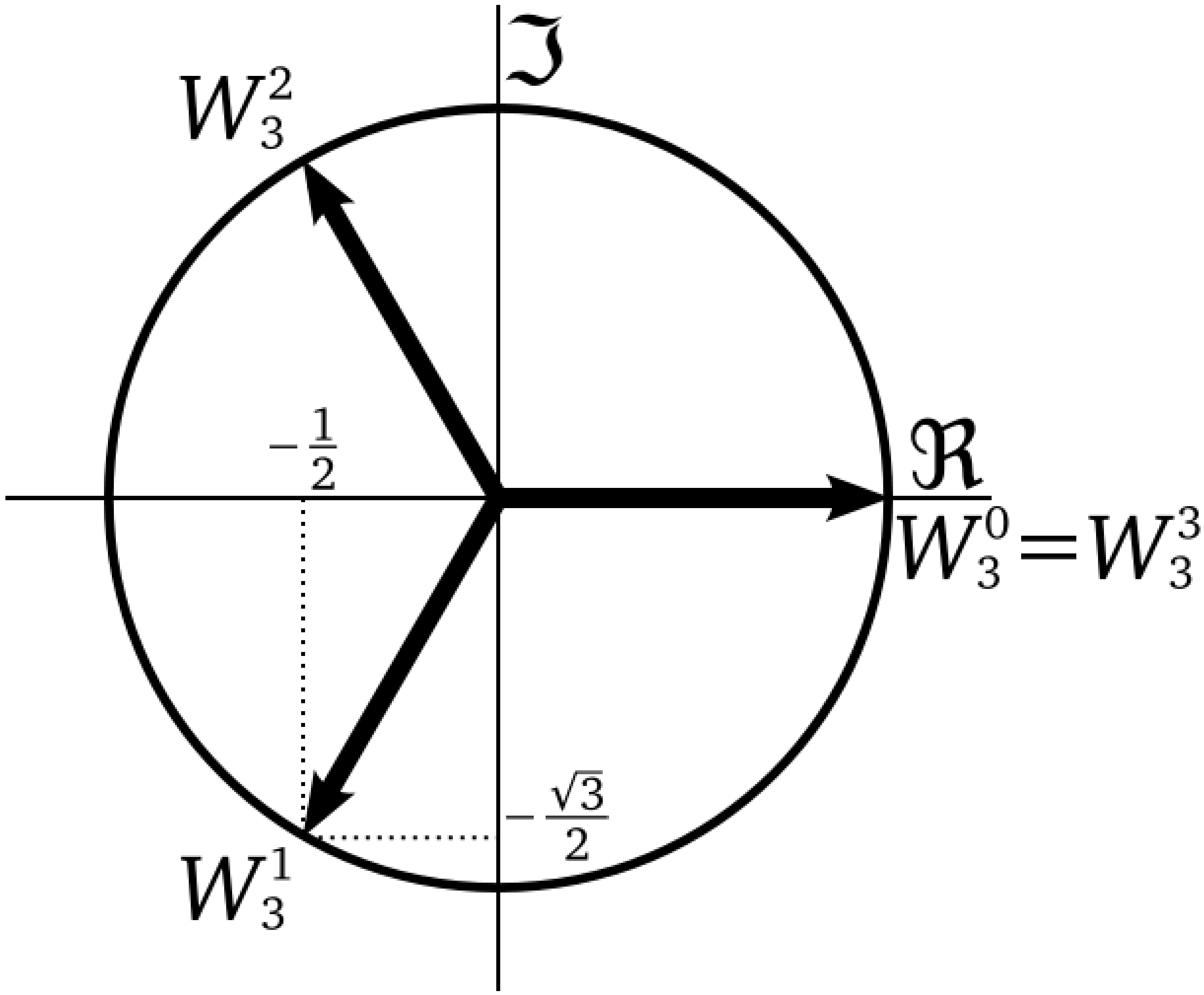
Why is it that when we consistently take the principal root we end up with 1, and if we consistently take the non-principal root we end up with:

$$-.5 \pm \frac{\sqrt{3}}{2} i$$

Epiphany: These values are the cube root of unity!

$$\sqrt[3]{1} = 1, -.5 \pm \frac{\sqrt{3}}{2} i$$

This is a graph showing the cube root of 1.



So as we continue to take the square root an infinite amount of times, we can determine the cube root. We just need to always take the *principal root* or the *non-principal root* an infinite number of times.

Let's review what the experts say:

Official Formula for Nested Radical: $\sqrt[n-1]{x} = x\sqrt[n]{x\sqrt[n]{x\sqrt[n]{x\sqrt[n]{\cdots}}}}$

Wait…this doesn't seem to be true. It seems that we have quite the opposite result! In fact, our results more closely follow this:

$$\sqrt[n+1]{x} = x\sqrt[n]{x\sqrt[n]{x\sqrt[n]{x\sqrt[n]{\cdots}}}} \text{ when } x = 1 \text{ and } n = 2$$

It seems like we became excited too fast. As we look at higher order roots we will notice an interesting behavior. The first row below summarizes the results if you take the principal root each iteration. Each additional row is another possible root that can be taken (if not blank). This table summarizes the results of constant iterations using the specified root.

Infinite Root Results using Different Root to Branch

Root Choice	$\sqrt[2]{\sqrt[2]{\sqrt[2]{\cdots}}}$	$\sqrt[4]{\sqrt[4]{\sqrt[4]{\cdots}}}$	$\sqrt[8]{\sqrt[8]{\sqrt[8]{\cdots}}}$	$\sqrt[16]{\sqrt[16]{\sqrt[16]{\cdots}}}$ *
1 (principal root)	$\sqrt[1]{1}$	$\sqrt[3]{1}$	$\sqrt[7]{1}$	$\sqrt[15]{1}$
i	-	$\sqrt[3]{1}$	$\sqrt[7]{1}$	$\sqrt[15]{1}$
$-i$	-	$\sqrt[3]{1}$	$\sqrt[7]{1}$	$\sqrt[15]{1}$
$\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}$	-	-	$\sqrt[7]{1}$	$\sqrt[15]{1}$
$\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}$	-	-	$\sqrt[7]{1}$	$\sqrt[15]{1}$
$-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}$	-	-	$\sqrt[7]{1}$	$\sqrt[15]{1}$
$-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}$	-	-	$\sqrt[7]{1}$	$\sqrt[15]{1}$
-1	$\sqrt[3]{1}$ (oscillating)	$\sqrt[5]{1}$ (oscillating)	$\sqrt[9]{1}$ (oscillating)	$\sqrt[17]{1}$ (oscillating)

* we have omitted some solutions for the 16th root, but the results are all the 15th root of unity

So, we were wrong to think that the negative root was part of any solution. Actually, the negative root seems to be the "odd-man out" while all the other roots converge to some number. This is definitely worth some more investigation.

In other words...

$$\sqrt[4]{\sqrt[4]{\sqrt[4]{\cdots}}} = \sqrt[3]{1} \text{...normal behavior, follows formula}$$

$$i\sqrt[4]{i\sqrt[4]{i\sqrt[4]{\cdots}}}=\sqrt[3]{1}$$ ...normal behavior, follows formula

$$-i\sqrt[4]{-i\sqrt[4]{-i\sqrt[4]{\cdots}}}=\sqrt[3]{1}$$ ...normal behavior, follows formula

$$-\sqrt[4]{-\sqrt[4]{-\sqrt[4]{\cdots}}}=?$$ ...**not normal**

Since the last root (-1) doesn't converge, it is commonly ignored, but this behavior is worth noting and we will come back to it.

Question: Why is the behavior at -1 so asymmetrical?

To continue studying this behavior we will need to know a little more about similar nested radicals.

Official Formula for Nested Radicals:
Officially, these *nested radicals* can be solved using the following formula:

$$\sqrt{n+\sqrt{n+\sqrt{\ldots n+\sqrt{n+\sqrt{\cdots}}}}}=x$$ $x^2-x-n=0$ for $n>0$

$$\sqrt{n-\sqrt{n-\sqrt{\ldots n-\sqrt{n-\sqrt{\cdots}}}}}=x$$ $x^2+x-n=0$ for $n>0$ **and** x is rational

We are essentially studying the case where **n does equal zero** and there is an additional negative on the outside of the outermost square root.

The rule for cube roots continues in a similar fashion:

$$\sqrt[3]{n-\sqrt[3]{n-\sqrt[3]{n-\sqrt[3]{n-\sqrt[3]{\cdots}}}}}=x$$ $x^3+x-n=0$ for $n>0$

Nested radicals using addition is fairly straightforward and doesn't yield too many surprises. The following table summaries some solutions. Almost every well-known transcendental number you've ever heard of can be written in the form of a nested radical.

For example, the table below identifies the Golden Ratio and the Plastic Constant.

n	$\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{n+\dots}}}}$	$\sqrt[3]{n+\sqrt[3]{n+\sqrt[3]{n+\sqrt[3]{n+\dots}}}}$	$\sqrt[4]{n+\sqrt[4]{n+\sqrt[4]{n+\sqrt[4]{n+\dots}}}}$
3	2.302775637731994...	1.671699881657	$x^4=x+3$ 1.45263...
2	2	1.5213797	$x^4=x+2$ 1.35321...
1	$\phi=1.61803399...$	$\rho=1.32471795724...$	$x^4=x+1$ 1.220744...
0	1	1	1
Area of Figure	4.112398179844034		

But, mathematicians don't want you to set n to zero because then you get:

$$x^2-x=0 \text{ when } n=0$$

$$x=1,0$$

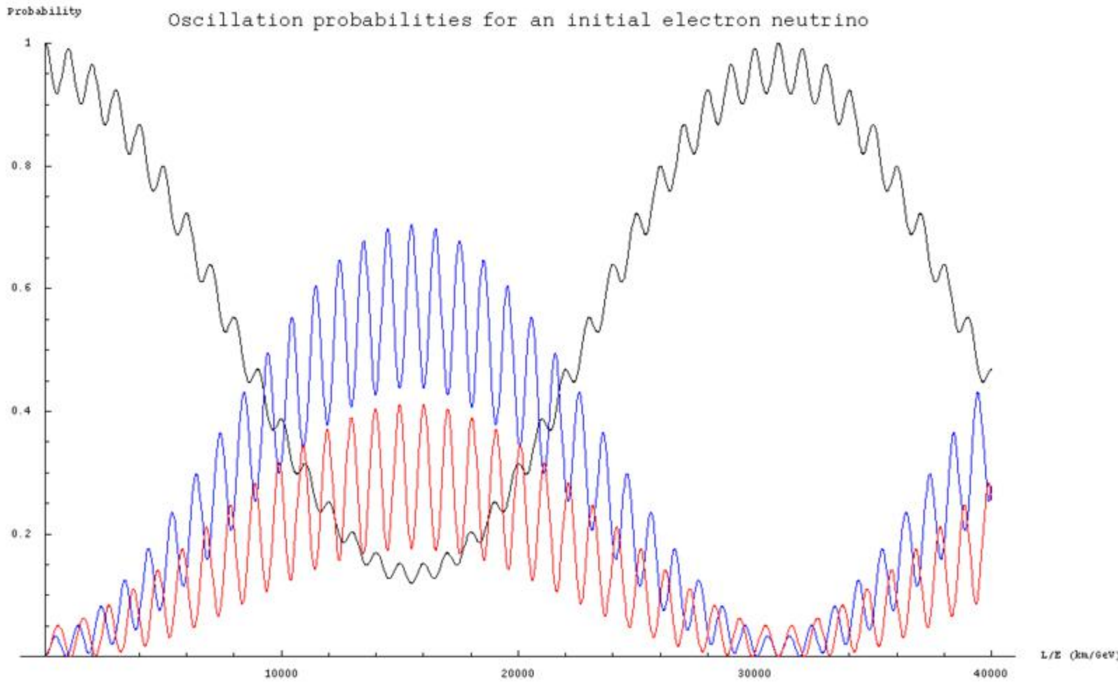
Meaning that from an infinite amount of square roots of zero, both zero **and** 1 are both logical solutions. This is a cardinal sin in mathematics. Why?

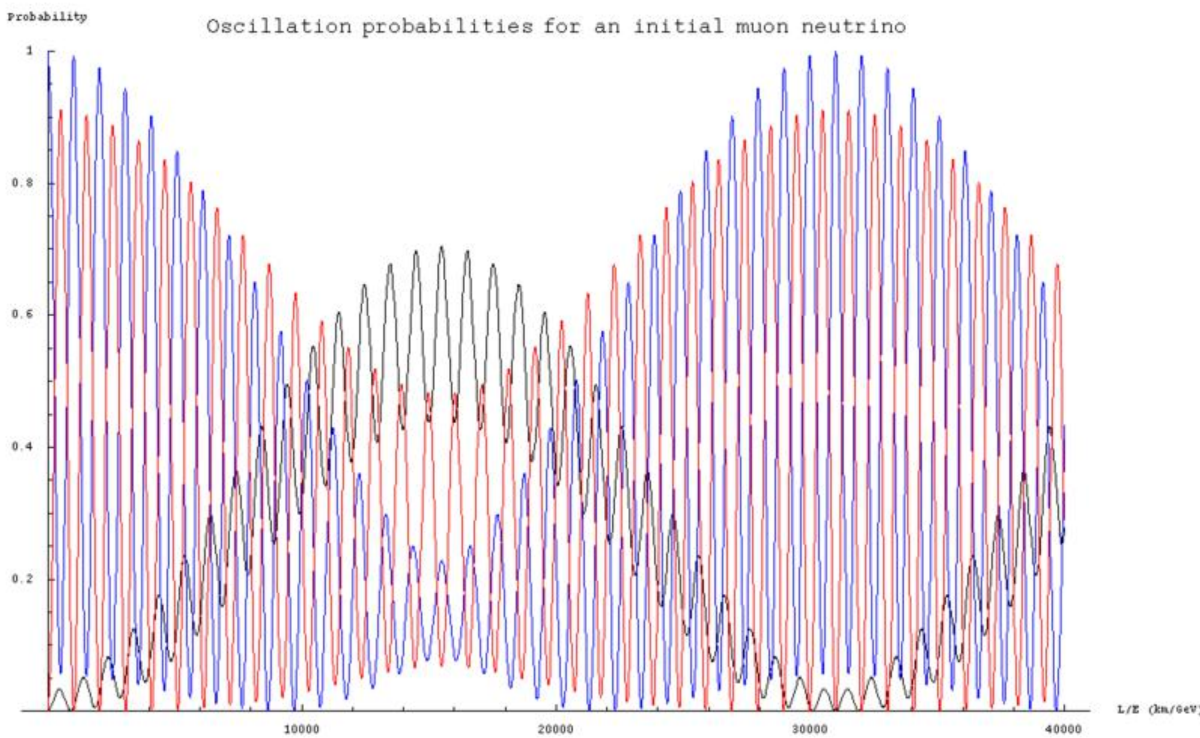
At the very center of mathematics there are *axioms*. An *axiom* is **not proven** but assumed to be self-evident. One of the axioms states that zero does not equal one. The above would violate that axiom!*

* Perhaps violate is the wrong word and I am using it for effect. It would indeed be difficult to come up with some intuitive reasoning why an infinite number of square roots equal both 1 and 0. Certainly that can't happen in nature, can it?

Neutrino Oscillations: Going from 0 to 1, and back again...

Neutrinos are the smallest particles known. Relative to other particles, they are virtually massless. In a way, they could be interpreted as nature's version of Zero. The interesting characteristic about neutrinos is their ability to oscillate between one form and another. Sometimes they are "nothing" (i.e. electron neutrino), and other times they are "something" (i.e. tau neutrino).





Newspaper Clipping: CERN Press Release

Geneva 13 May, 2010

Several experiments since have observed the disappearance of muon-neutrinos, confirming the oscillation hypothesis, but until now no observations of the appearance of a tau-neutrino in a pure muon-neutrino beam have been observed: this is the first time that the neutrino chameleon has been caught in the act of changing from muon-type to tau-type.

Source: http://press.web.cern.ch/press/PressReleases/Releases2010/PR08.10E.html

When Math Changes its Rules

Referring to *Official Formula for Nested Radicals*, I get a vastly different solution when n=0. In a smooth contiguous world, should these two solutions be equal as n approaches zero? Why would we use one formula when n gets very small, but then be forced to have a different formula when n becomes zero? It seems like there are too many exceptions in modern mathematics...

So, when n is any number except zero there is a formula as described above:

$$x^2 - x - n = 0 \text{ for } n > 0$$

But, when n=0, our new formula resembles something like* this:

$$x^3 - 1 = 0$$

*I use the word *like* because it is an initial theory. However, we will see that sometimes we need to modify a theory with more information.

Clearly, we must inspect this more deeply. When we incorporate subtraction at each iteration, we open a whole realm of mystery. If this is indeed true, we must examine exactly where it is that mathematics breaks down and at what point do our formulas need to be modified.

Example: Iterative Method versus a Formula

We will be using this *Official Formula for Nested Radicals* when we refer to "the Formula".

$$\sqrt{n - \sqrt{n - \sqrt{...n - \sqrt{n - \sqrt{...}}}}} = x$$

$$x^2 + x - n = 0 \text{ for } n > 0 \text{ and x is rational}$$

When we perform the "Iterative Method", we are manually performing each square root and multiplying the result by negative one, over and over…We hope to compare these results to see if "the Formula" always holds true, at every point.

To see our first instance of where mathematics breaks, down, we will examine numbers smaller than 1. One the solution becomes irrational, the formula no longer holds. We will learn that a new formula must be used when our solutions become irrational.

More Examples: Iterative Method versus Formula (Skip if you Hate Formulas)

Mathematics becomes even more interesting when we n<1...

Formulaic Approach: $x=\sqrt{.9-x}$

Iterative Approach: $\sqrt{.9-\sqrt{.9-\sqrt{.9-\sqrt{.9-...}}}}$

Are these equal?

Formula Approach
Solving the Formulaic Approach:

$$x^2+x-.9=0$$

$$x=0.572381,-1.57238$$

Iterative Approach
Continuous iterations of the above strictly taking one branch at each iteration (principal or non-principal):

$$x=0.572381,-1.57238$$

Answer: at n=.9, YES, both approaches yield the same result

How boring! The formula actually matches the result of doing the problem by hand. Let's see if we can't find something more interesting...

Do we concur at smaller numbers such as .7?

Equation 1: $x=\sqrt{.7-x}$

Equation 2: $\sqrt{.7-\sqrt{.7-\sqrt{.7-\sqrt{.7-...}}}}$

The answer is NO.

Formula Approach
Solving the Formulaic Approach:

$$x^2+x-.7=0$$

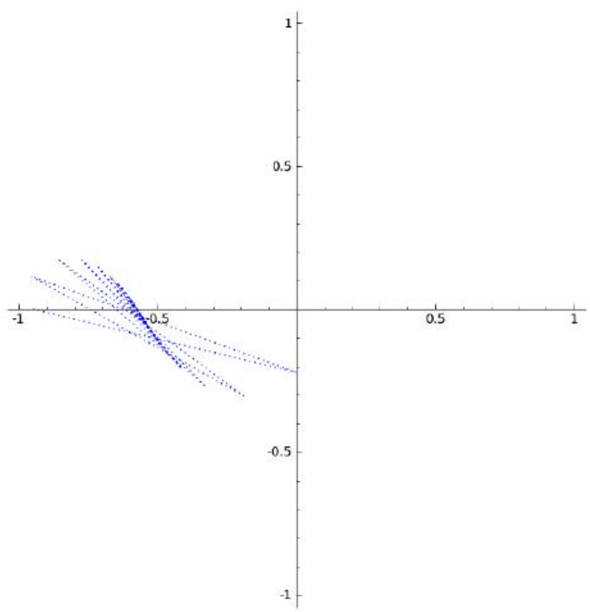
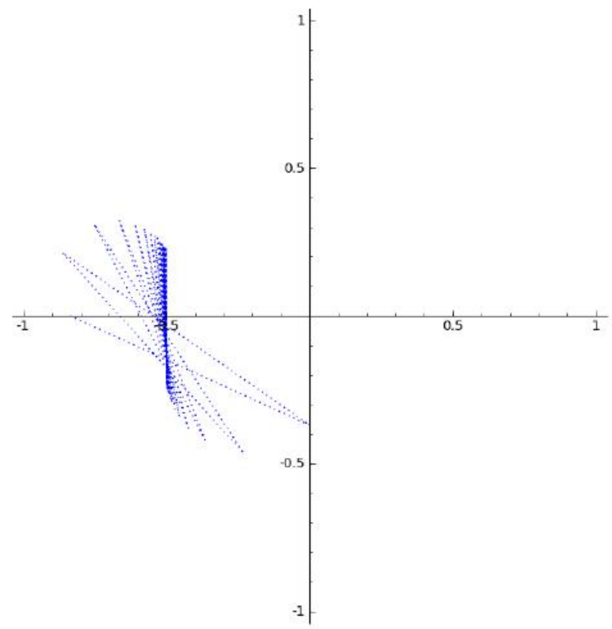
$$x=0.474679,-1.47468$$

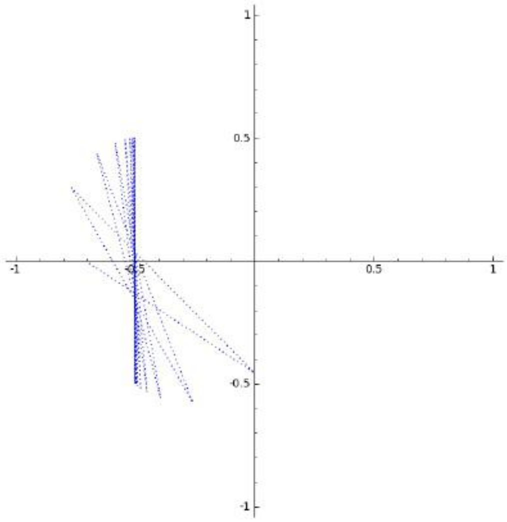
Iterative Approach
Continuous iterations of the above strictly taking one branch at each iteration (principal or non-principal):

Solving iteratively gives us a solution with real part **.5**, and an imaginary part that oscillates. The iterative method never reaches the theoretical solution, the solution that the formula describes!

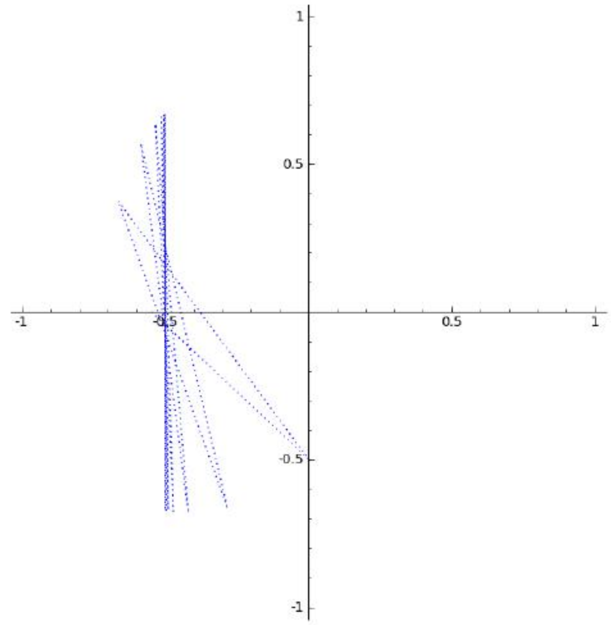
The graphs below illustrate the iterative method solutions to each problem (taking just the principal root).

Iterative Method for Various Additive Nested Radicals Smaller than 1

$-\sqrt{.9-\sqrt{.9-\sqrt{.9-...}}}$	$-\sqrt{.7-\sqrt{.7-\sqrt{.7-...}}}$
 This result completely concurs with the *Official Formula* for Nested Radicals (x=-.572381...).	 The result constantly oscillates and **does not** concur with the *Official Formula* for Nested Radicals.
$-\sqrt{.5-\sqrt{.5-\sqrt{.5-...}}}$	$-\sqrt{.3-\sqrt{.3-\sqrt{.3-...}}}$

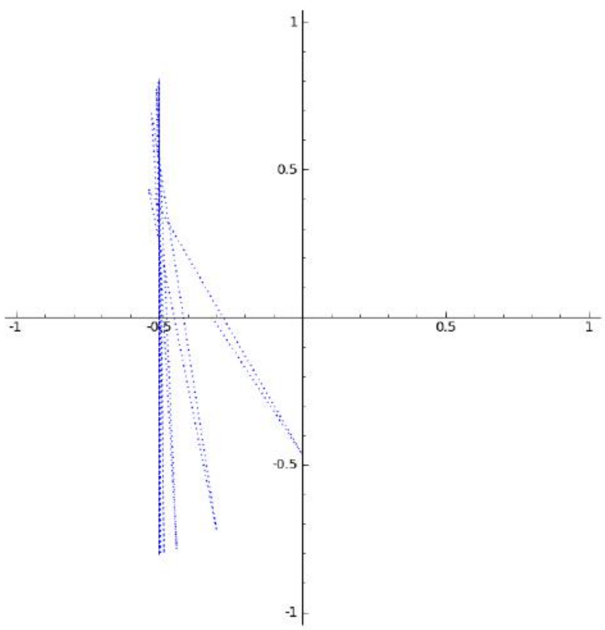


The solution oscillates at $-.5 \pm .5i$.



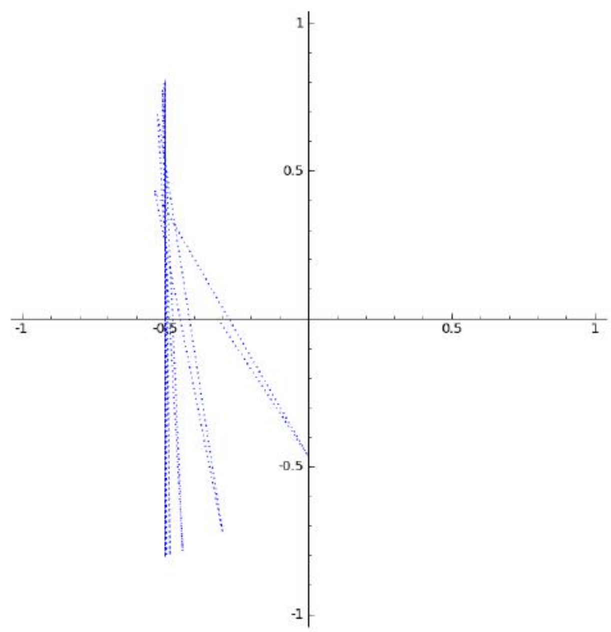
$$-\sqrt{.1-\sqrt{.1-\sqrt{.1-\dots}}}$$

$$-\sqrt{0-\sqrt{0-\sqrt{0-\dots}}}$$



It's like the function hit a brick wall and began to flatten out like a pancake.

$$-.5 \pm \frac{\sqrt{3}}{2} i$$



You will notice that this concurs with our study of

$$-\sqrt{-\sqrt{-\sqrt{-\sqrt{\dots}}}}$$

At what value do the formulas and the iterative solutions diverge?

It seems that the real part of x becomes permanently stuck when $n = \frac{3}{4}$.

After 800 iterations, $-\sqrt{.75-\sqrt{.75-\sqrt{.75-\sqrt{.75-...}}}}$

[-0.50025592823677933473902612604, -0.02534458549264308191428796 1354],

[-0.50038528253554404034986191618, 0.02532507087760197186931954 1497], [-
0.50025535687186084565780674893, -0.02531214361797321649172332 0587], [-
0.50038421657678955334347392636, 0.02529270786270772231052461 6205], [-
0.50025478810199852652829122235, -0.02527982586500572559362609 5815], [-
0.50038315635536845249475845776, 0.02526046844695565077310028 8120], [-
0.50025422190942503419819331612, -0.02524763144480693041749042 0382], [-
0.50038210182564601957722061343, 0.02522835184620998358329124 7555], [-
0.50025365827653545114372946045, -0.02521555957544242138057555 2803], [-
0.50038105294246626453258386391, 0.02519635728327798002146309 3863], [-
0.50025309718588495400126703316, -0.02518360948189739267899511 7876], [-
0.50038000966114626333336445896, 0.02516448398783112036514175 7134]]

Even if we attempt $-\sqrt{.7-\sqrt{.7-\sqrt{.7-\sqrt{.7-...}}}}$, we still are in an endless loop with real part $-\frac{1}{2}$.

It seems like we need to use the following formula:

$$-\sqrt{n-\sqrt{n-\sqrt{...n-\sqrt{n-\sqrt{...}}}}} = x$$

$$(-x)^2 + (-x) - n = 0 \text{ for } n > .75$$

$$x^2 + x - n + 1 = 0 \text{ for } .75 > n > 0$$

For example:

$$-\sqrt{.9-\sqrt{.9-\sqrt{....9-\sqrt{.9-\sqrt{...}}}}} = x \Rightarrow (-x)^2 + (-x) - .9 = 0$$

$$-\sqrt{.7-\sqrt{.7-\sqrt{....7-\sqrt{.7-\sqrt{...}}}}} = x \Rightarrow x^2 + x - .7 + 1 = 0$$

But why?

So a more official rule could be:

Our rule for Nested Radicals:

$$-\sqrt{n-\sqrt{n-\sqrt{...n-\sqrt{n-\sqrt{...}}}}}=x$$

$(-x)^2-x-n=0$ for $n>.75$
$x^2+x-n+1=0$ for $.75>n\geq 0$

and for the non-principal root $x^2-x-n=0$ for $n\geq 0$

In general, it is important to know that our *rules had to change when we reached the border of the real versus imaginary solutions. Our entire formula had to change.*

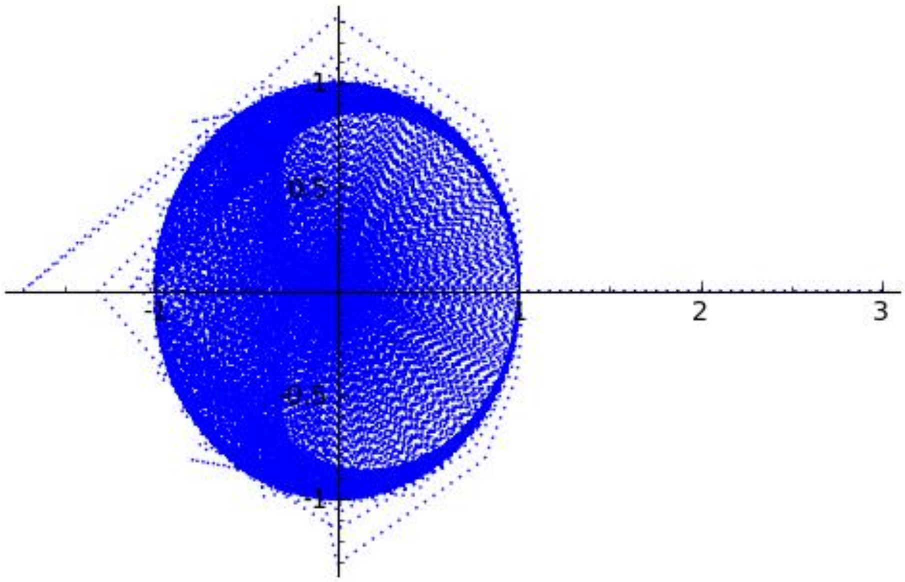
We just saw how we need to use different rules and formulas when we work with problems containing only real solutions to problems including imaginary numbers. We will see in Chapter 5 that we must take another leap and use a different set of rules and formulas for zero and infinity.

Square Roots: Branching in Both Directions

But, what if we don't take the principal root each time. With each iteration, we could pick one of two directions; what happens if we could go both directions? So, for each branch, we could take both the principal and non-principal root and branch out indefinitely. What will happen?

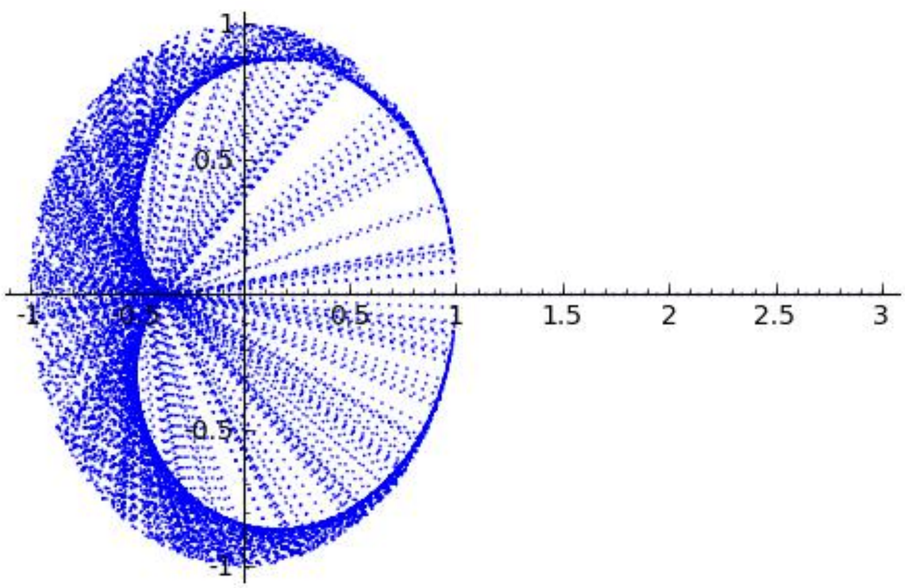
Iterations of the Square Root of 3 (Using both roots at each iteration)

$$-\sqrt{-\sqrt{-\sqrt{-\sqrt{...\sqrt{3}}}}}$$

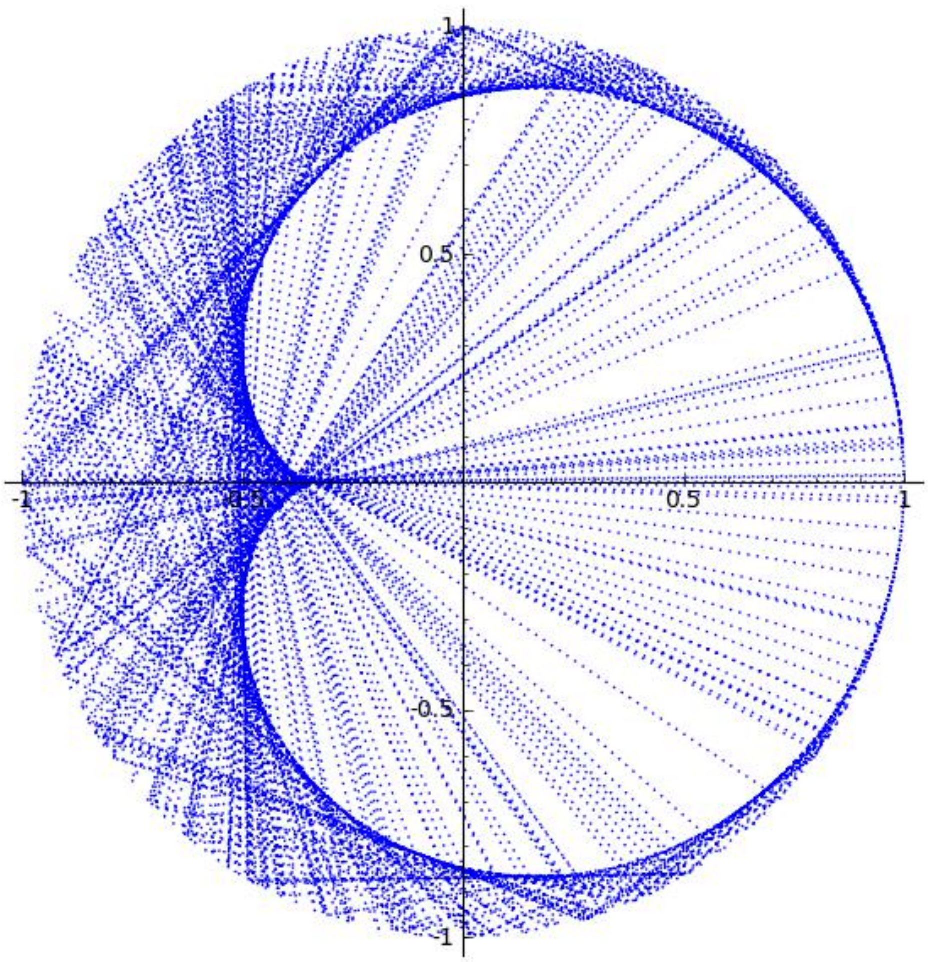


You will see that the solutions works out almost perfectly if we randomly pick the principal root or the non-principal root. Let's explore some iterations, but instead of graphing both possible paths, let's randomly pick just one direction at each opportunity. We will pick either path without bias by setting a 50% probability of selecting a given path.

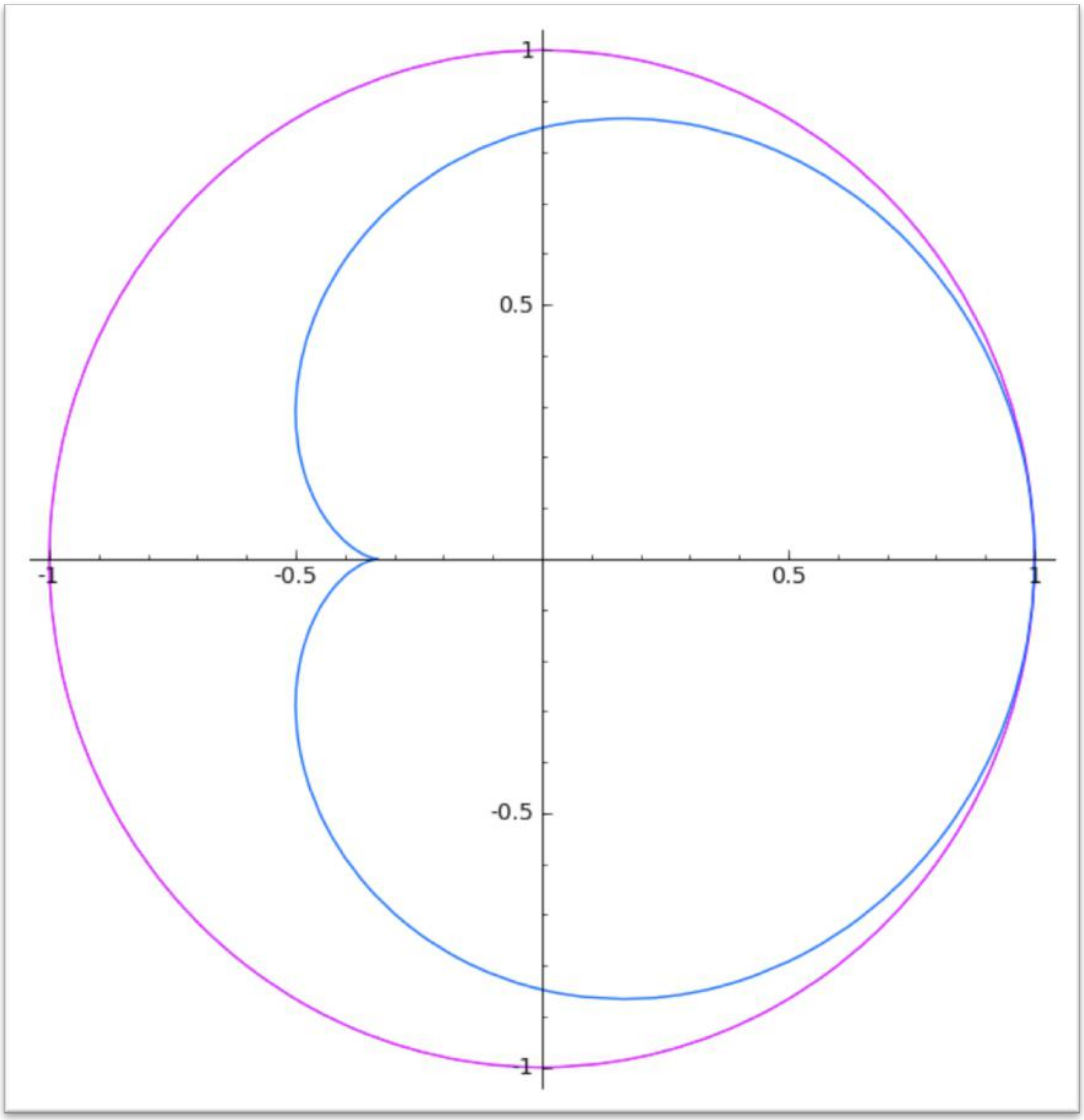
Iterations of Square Root of 3 using a random root (50% principal/50% non-principal)



Zooming in on the central figure...



You will notice a shape forming inside the object. Formally, this shape is called a *cardioid*. This heart-like shape can be created by tracing a point on the perimeter of a circle that is rolling around a fixed circle of the same radius.



The sage code used to generate the above chart is below.

Sage Snippet:
```
p1=parametric_plot([cos(x),sin(x)],(x,0,2*pi),rgbcolor=hue(0.8))
p2=parametric_plot([(-1/3)-(2/3)*cos(x)*(1-cos(x)),(2/3)*sin(x)*(1-
cos(x))],(x,0,2*pi),rgbcolor=hue(0.6))
show(p1+p2,figsize=[8,8])
```

This particular cardioid has a vertex at -1/3 and area of: $\frac{2\pi}{3}$.

Link to Nature

When light hits the rim of a cup at the correct angle, the image created is eerily similar to what we have just created using only the most basic of mathematical operations.

Photo by Gérard Janot

(licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license)

The cardioid can be found in countless other places in nature and plays a key role in acoustics.

More Connections: The Mandelbrot Set

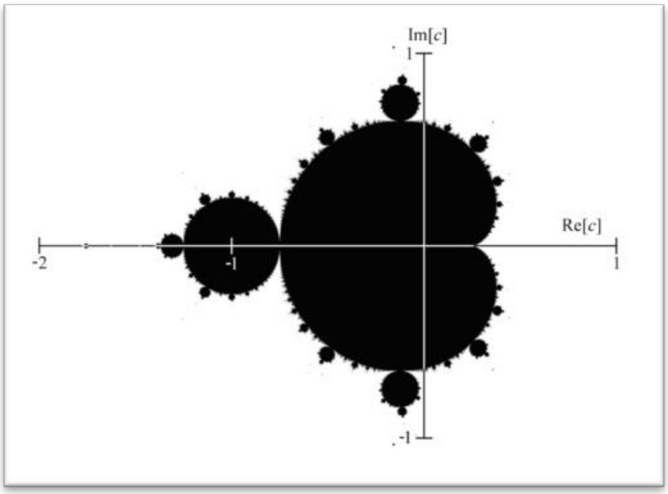
If you have ever picked up a book on Chaos Theory, you will immediately notice that this looks like a famous mathematical pattern within a Mandelbrot Set.

The Mandelbrot set is closely related to infinitely nested square roots with a slight modification.

Mathematically, the Mandelbrot set can be defined as the set of complex values of c for which the orbit of 0 under iteration of the complex quadratic polynomial $z_{n+1} = z_n^2 + c$ *remains bounded (Wikipedia).*

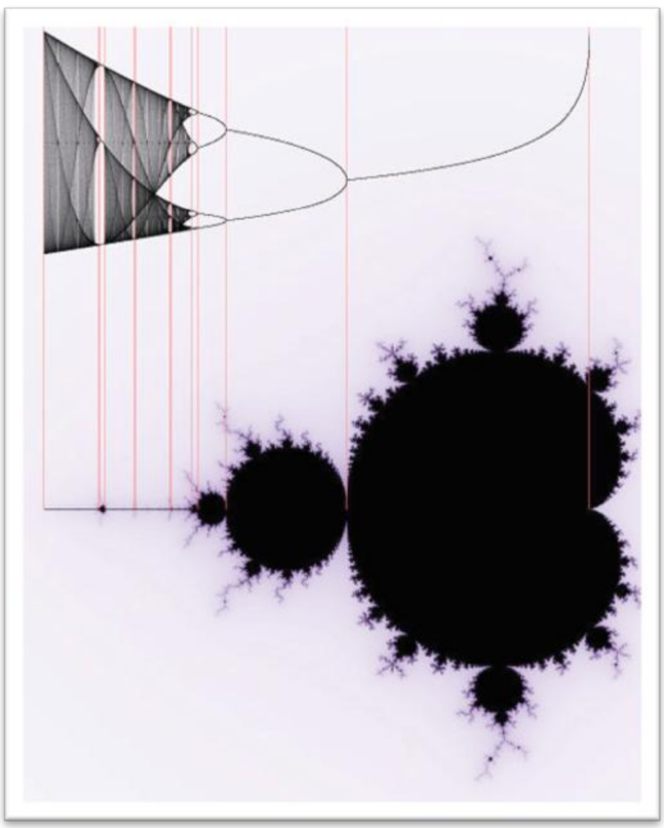
As one zooms inside this shape, strange patterns emerge. Shapes that resemble seashells, seahorses, galaxies, and more have all been spotted in this shape after intense magnification. The shape, spawned from such a simple function, appears to create infinitely complex objects.

It is believed to be transcendental but not yet proven.



Furthermore, the Mandelbrot set has been directly linked to two very important constants in a dynamic system, the Feigenbaum constants.

The following graph shows the relationship between the Mandelbrot set and a Bifurcation Diagram for Logistic Maps.



The key point is to notice that we have related many concepts and mathematical constants all with one simple experiment:

- ✓ Square Roots
- ✓ Imaginary Numbers

- ✓ Pi, and therefore e
- ✓ The concept of Zero and Infinity
- ✓ Randomness
- ✓ The Feigenbaum constants
- ✓ Light / Optics (to some degree)

How About Quad Roots?

Here is another way to take a look at infinite roots. Let's take -9 for example:

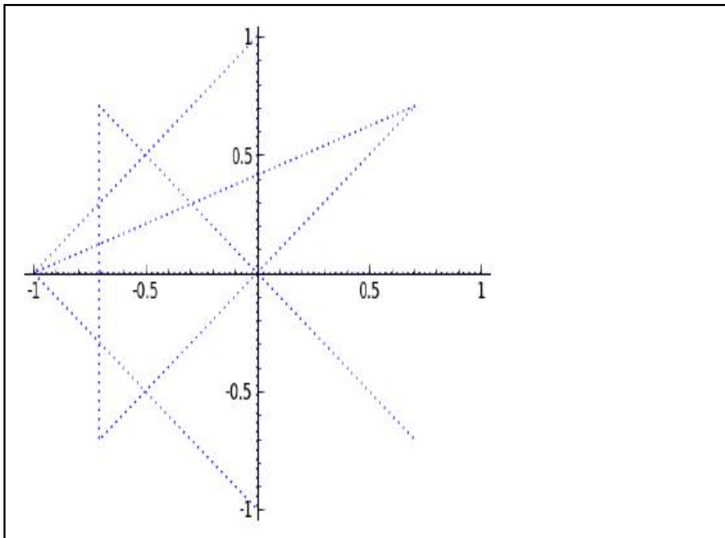
$$\sqrt[4]{-9} = \pm 1.22474 + 1.22474i, \pm i(1.22474 + 1.22474i)$$

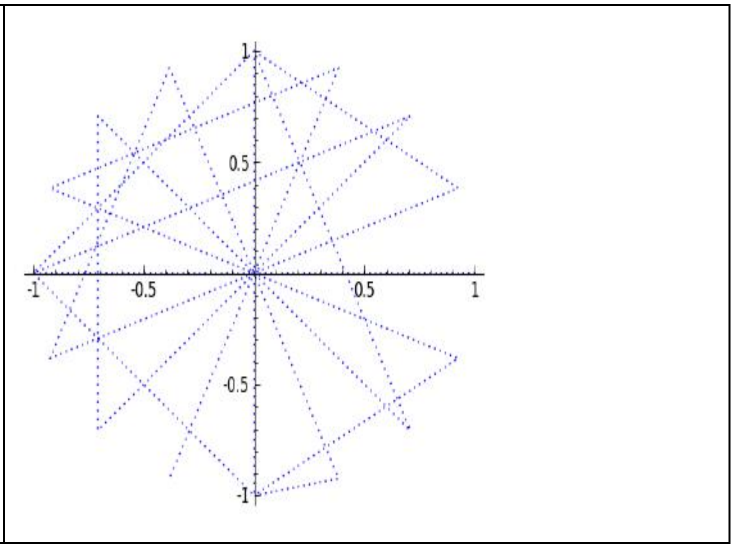
So each time you take a fourth root you can multiply the result by -1, 1, i, or –i to find all four unique solutions. If we start with any number and constantly calculate the fourth root, we start to notice a pattern. All points will eventually form a nice unit circle. However, if we draw a line from each result to the next, a very odd, asymmetry occurs similar to our branching with square roots.

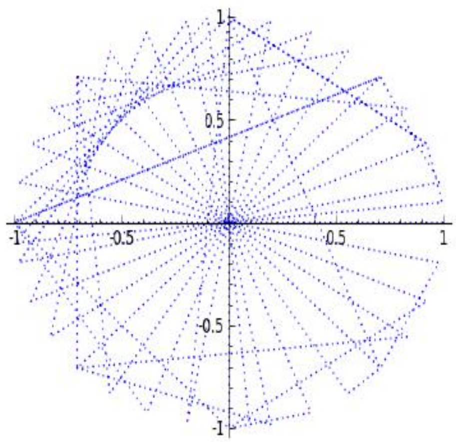
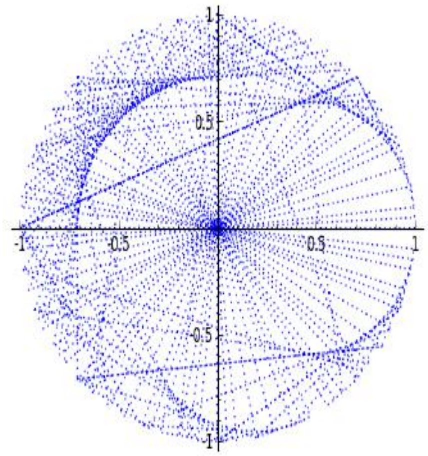
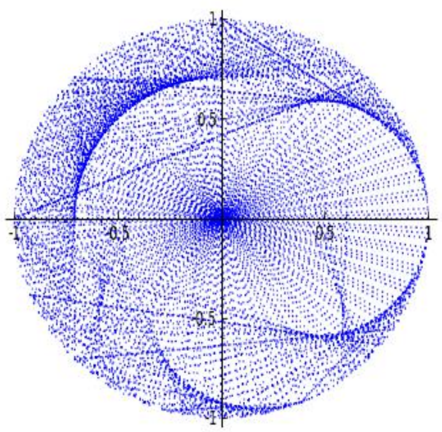
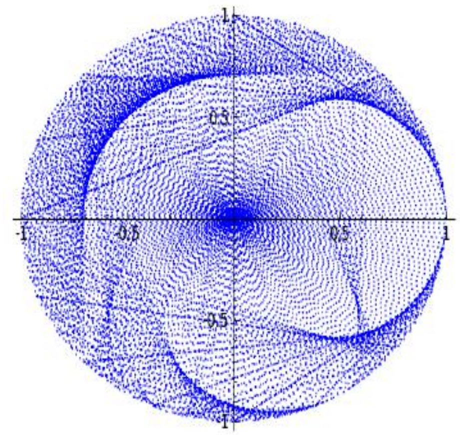
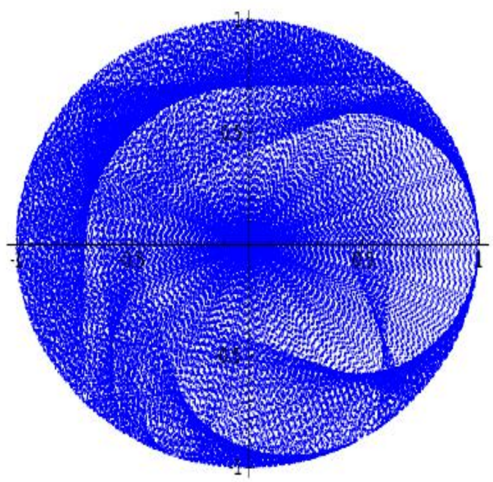
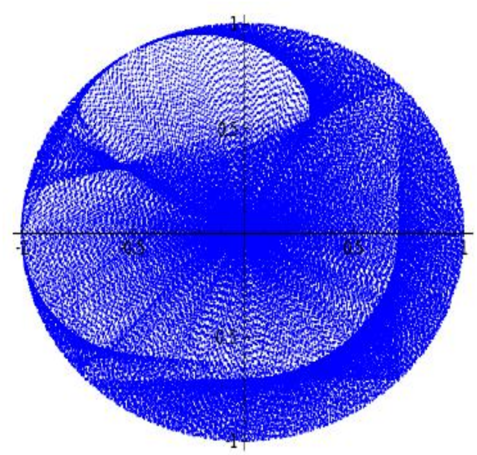
Iterations of the Fourth Root of 1(Using all four roots at each iteration)

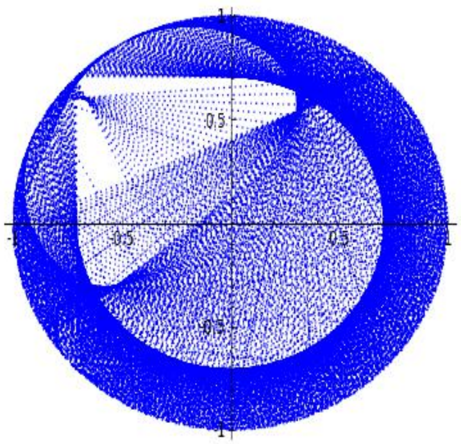
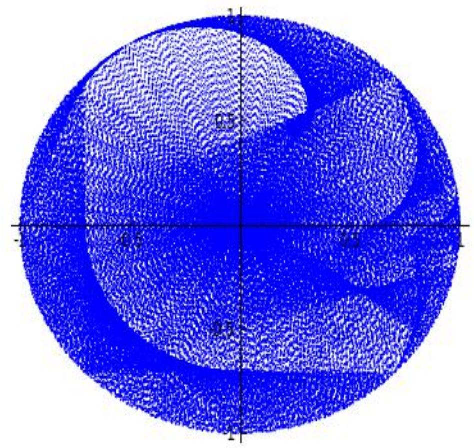
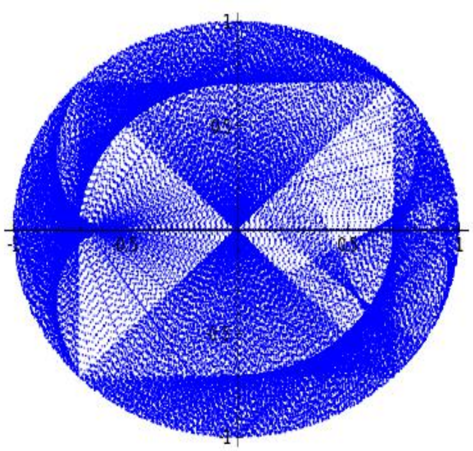
$$\sqrt[4]{\sqrt[4]{\sqrt[4]{\dots\sqrt[4]{\sqrt[4]{1}}}}}$$

Below illustrates the results of a simple recursive python program that attempts to graph the above function.





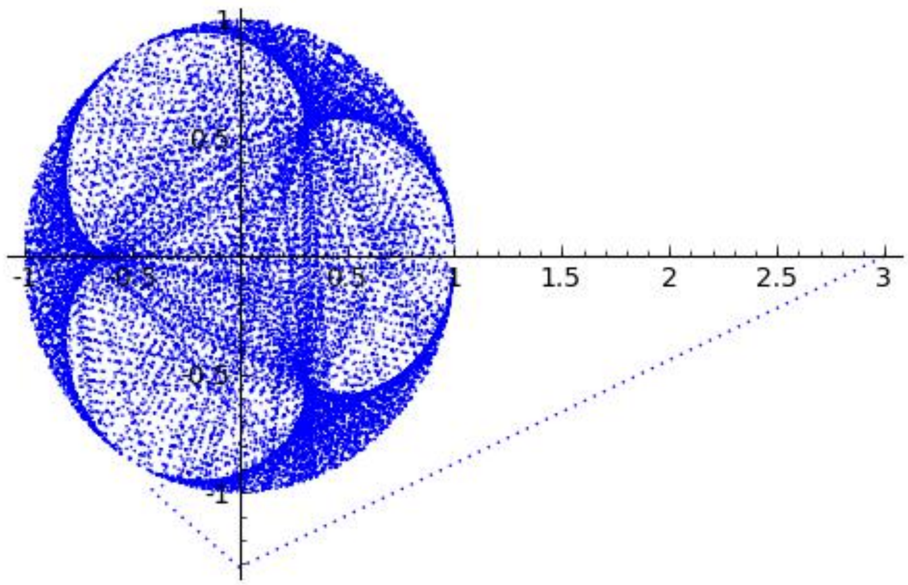
	
	
However, the lines drawn depend on what order we do our recursion. Here is one possible drawing: 	g(i*value^(1/4),array,j) g(-i*value^(1/4),array,j) g(value^(1/4),array,j) g(-1*value^(1/4),array,j) 
g(i*value^(1/4),array,j) g(value^(1/4),array,j) g(-i*value^(1/4),array,j)	g(i*value^(1/4),array,j) g(-i*value^(1/4),array,j) g(-1*value^(1/4),array,j)

g(-1*value^(1/4),array,j) 	g(value^(1/4),array,j) 
g(-1*value^(1/4),array,j) g(i*value^(1/4),array,j) g(-i*value^(1/4),array,j) g(value^(1/4),array,j) 	

Interestingly, no matter what order we seem to pick, we will always get an asymmetrical graph. This is partially because python is a *pass by reference* language in the case of nested recursion with integers. We can achieve order from this disorder by introducing a very odd element to the operation - **randomness**.

If we randomly decide which root we are using, a pattern emerges. In fact, we cannot achieve symmetry without randomness.

Iterations of the Fourth Root of 3 (Using a random roots at each iteration, 1, -1, i, or -i)



Notice the inner circles intersect the border of the imaginary circle at two of the crucial points: $-.5 \pm \frac{\sqrt{3}}{2} i$

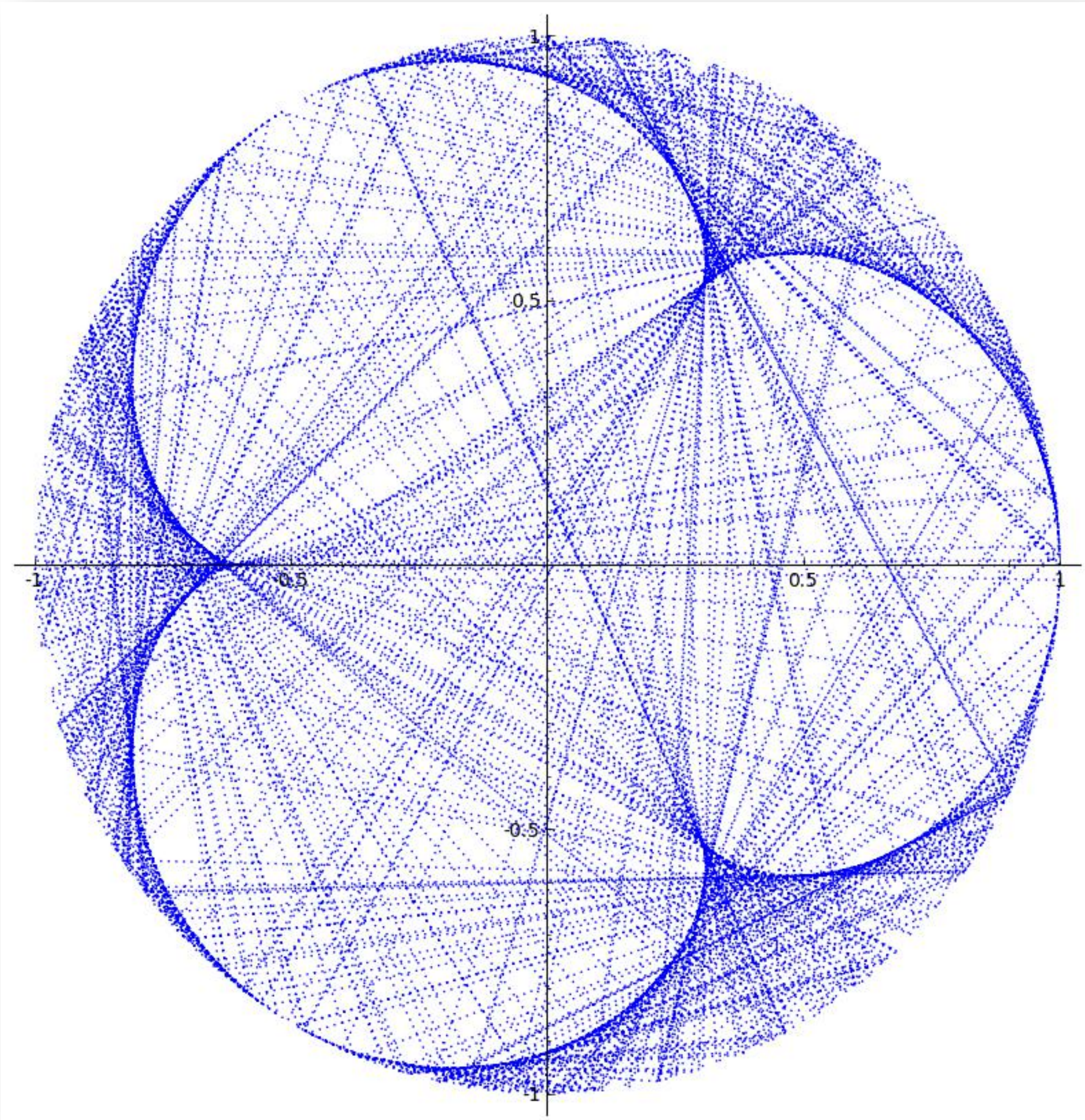
The inner circles have vertices at -3/5, and 3/10 +/- 1/2i

The resulting figure appears to be a three-leaf clover contained inside a circle. This is the result no matter what number, real or imaginary, we start with. After just a few hundred iterations, we can see the shape clearly defined.

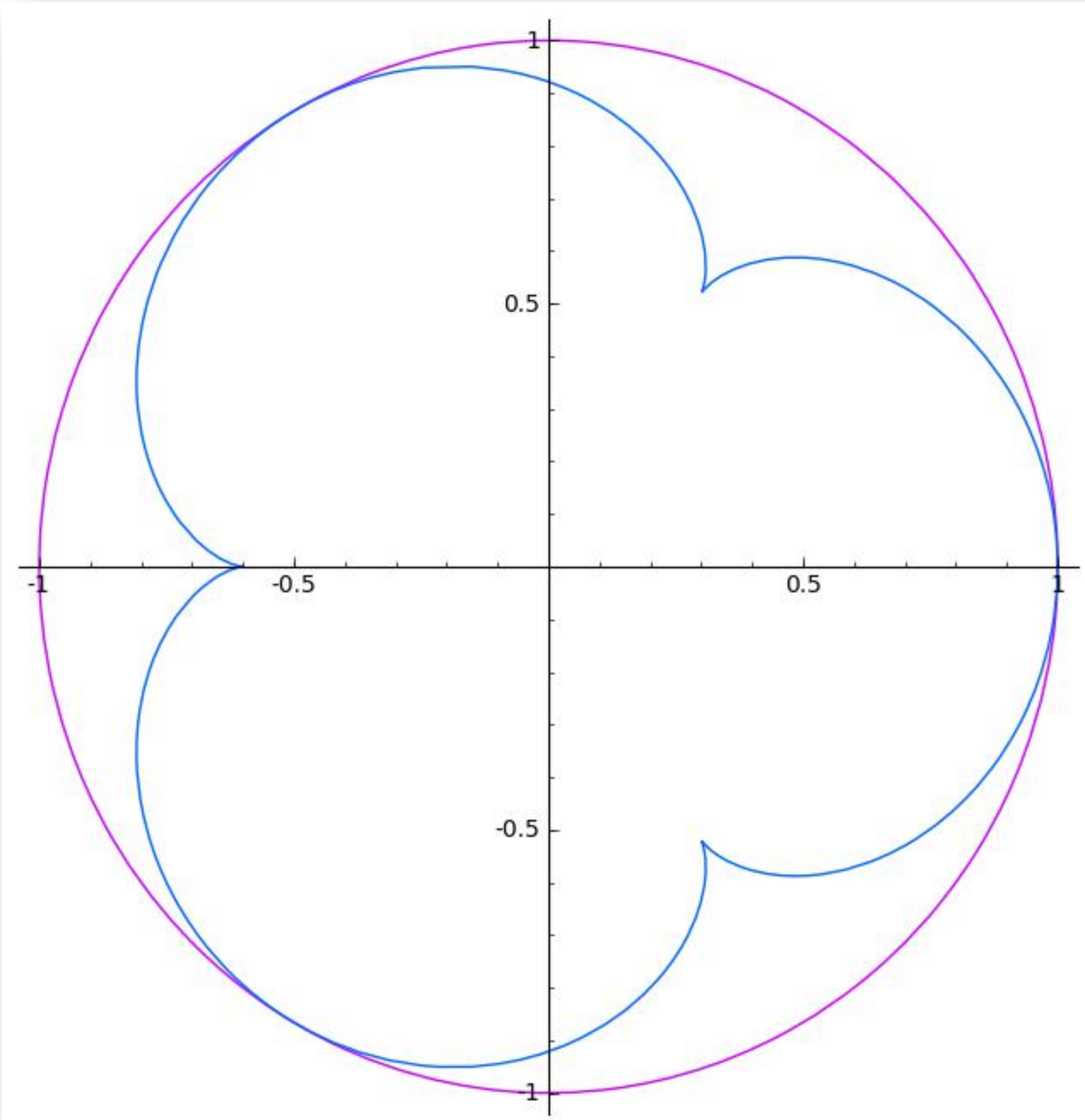
Since there are four roots, there are four different ways to branch, four possible choices. Each time we make a decision with equal probability. In other words, we choose the negative quad root with equal probability as taking the *ith* quad root, with equal probability of taking the negative *ith* quad root, and so forth.

No matter where we start, we always converge to the figure below. A line is drawn from one result to the other. The following graph is a high-resolution image of the key figure centered about the origin.

Iterations of the Fourth Root (Using a random roots at each iteration, 1, -1, i, or -i)



The inner figure can be graphed parametrically as we did earlier. We can then perform simple integration to calculate the inner area.



Sage Snippet:
```
p1=parametric_plot([cos(x),sin(x)],(x,0,2*pi),rgbcolor=hue(0.8))
p2=parametric_plot([-1/5*((3+1)*cos(x)-(cos((3+1)*x))),-1/5*((3+1)*sin(x)-
(sin((3+1)*x)))],(x,0,2*pi),rgbcolor=hue(0.6))
show(p1+p2,figsize=[8,8])
```

The Area can be calculated as follows:

```
g=-1/5*((3+1)*sin(x)-(sin((3+1)*x)))
```

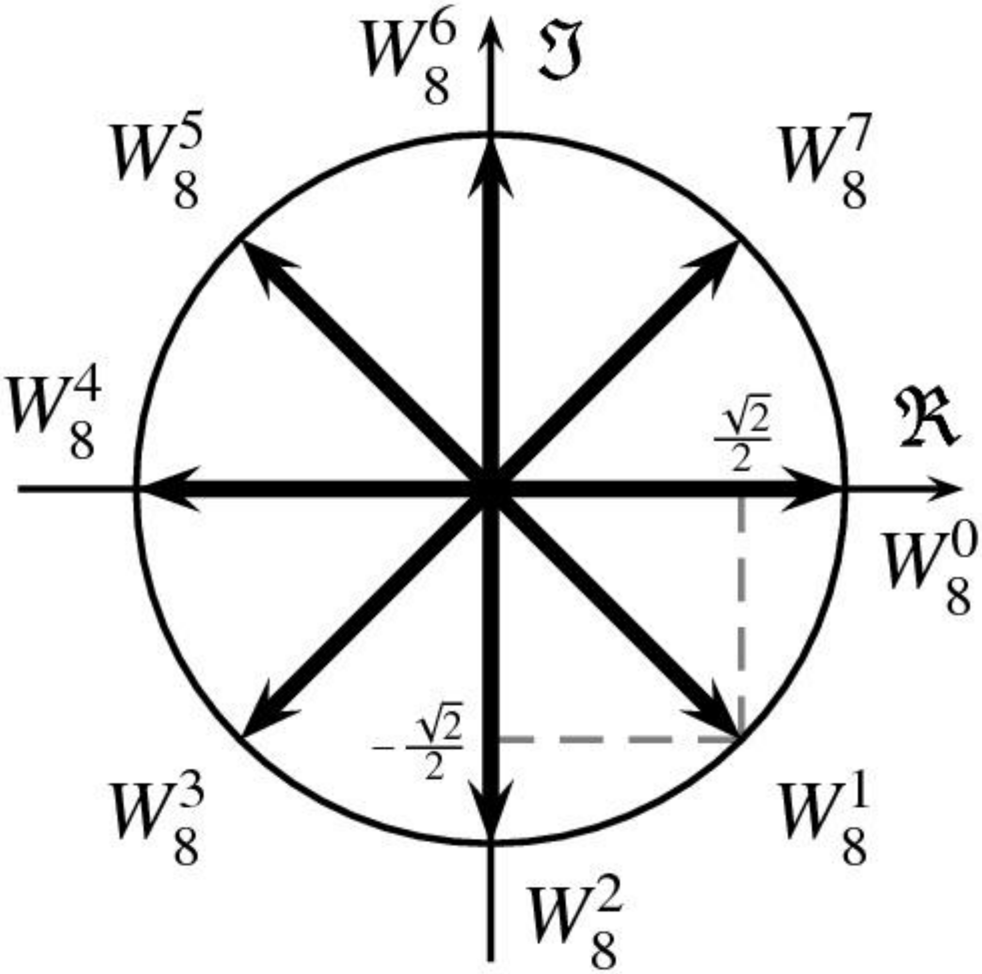
```
//print g.diff(x)
f=(-1/5*((3+1)*cos(x)-(cos((3+1)*x)))) * (4/5*cos(4*x) - 4/5*cos(x))
print f.integrate(x,0,2*pi)
```

This integration results in Area of $\frac{4\pi}{5}$.

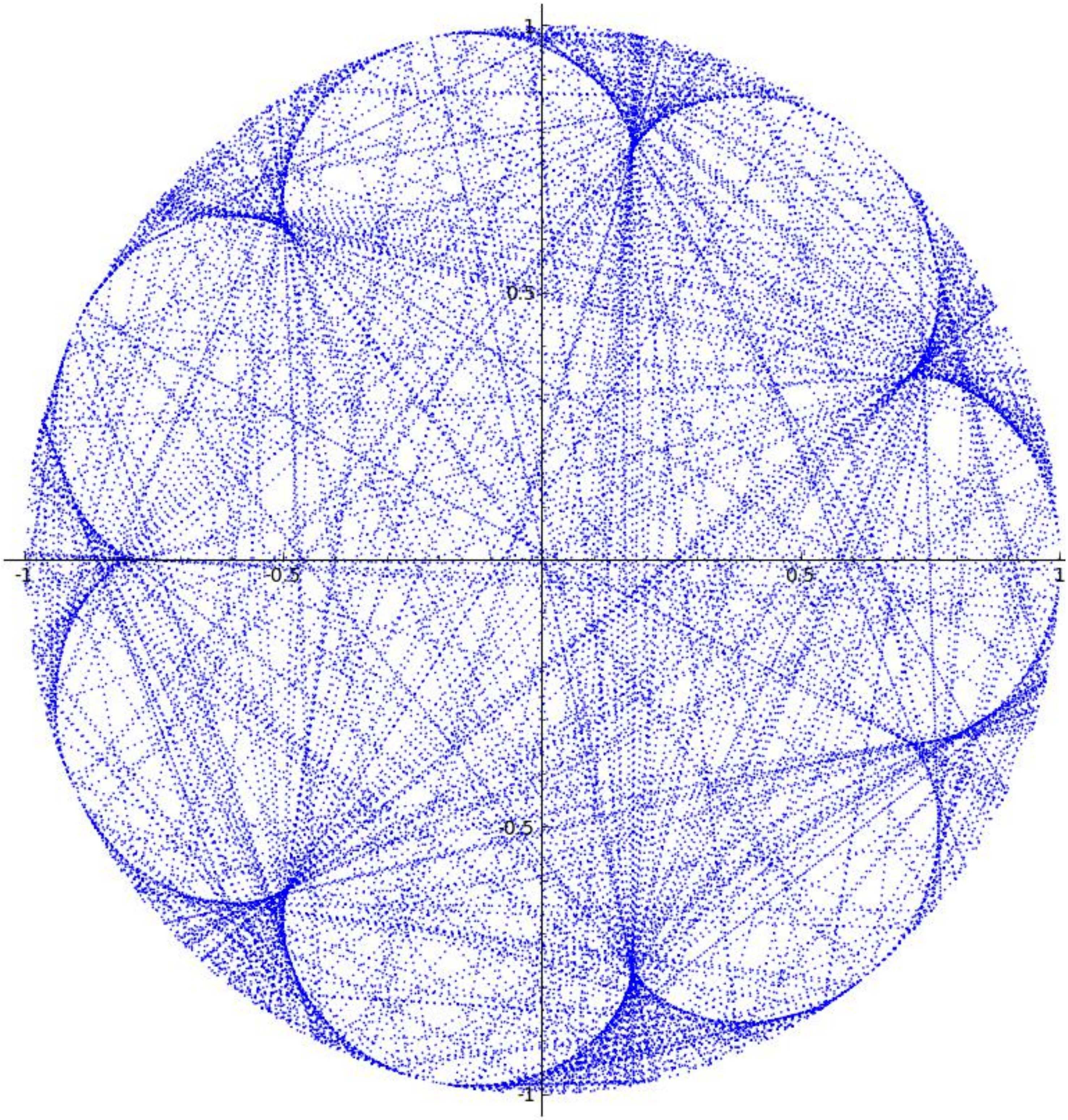
Eighth Root of Unity

Our next example will involve eight possible branches for each iteration. Rather than simply multiply the principal root by -1 or 1, we will multiply the principal root by our choice of eight possible values. These eight possible choices are called the eighth root of unity.

Drawn on a circle in the complex plane (real number on the horizontal axis and imaginary number on the vertical axis), you can see our eight choices.



Eighth Root of 1 (Randomly Branch)



You can see that this figure has seven leaves.

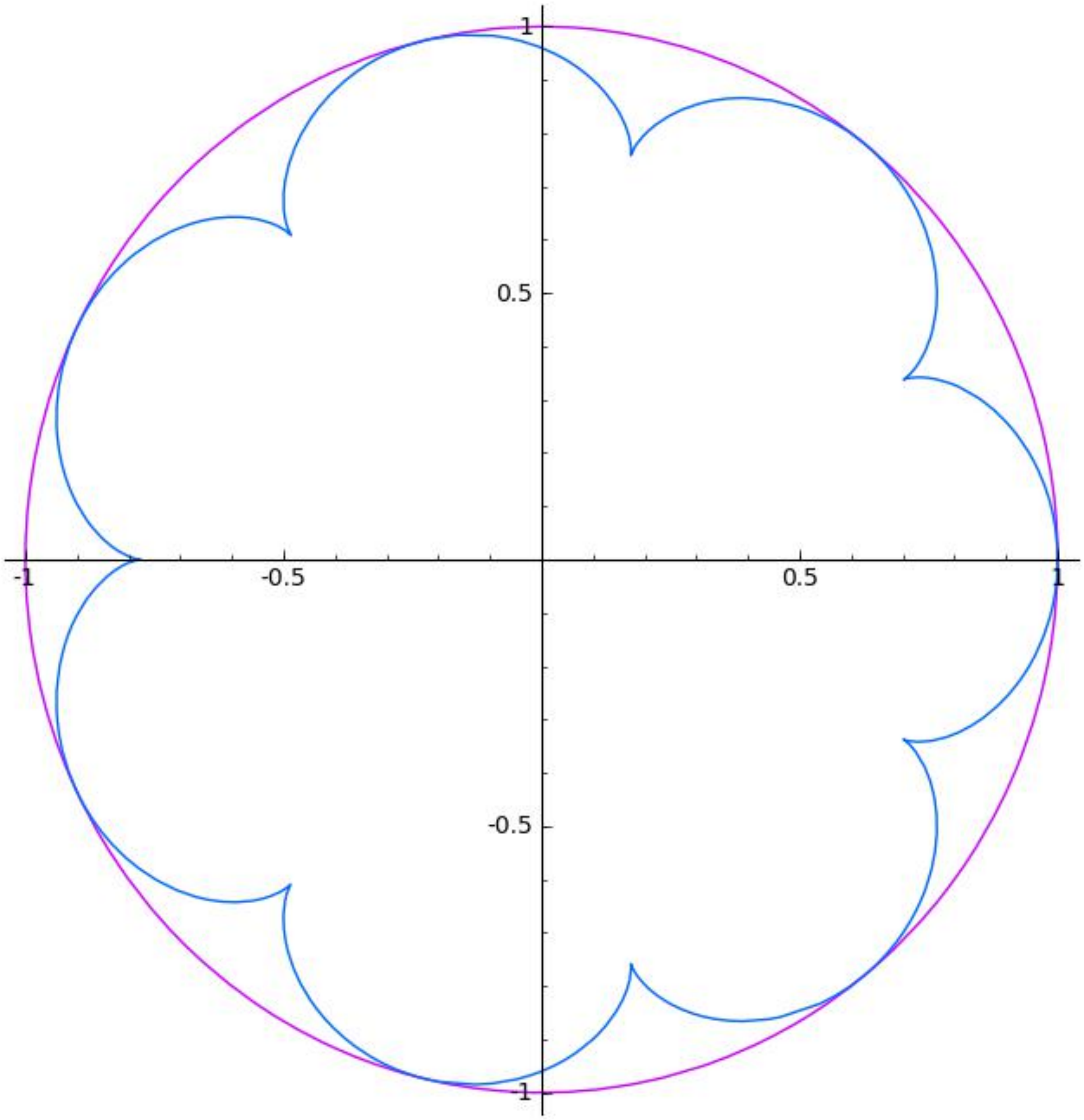
Its area can be calculated to be: $\frac{8\pi}{9}$.

You will notice that based on the below snippet of code, this graph was generated after only 500 random iterations. It is also important to note that we must use the negative root and every other root equally to produce a perfect graph.

Sage Snippet:

```
import random
def g(value,array,j):
   array.append([real(numerical_approx(value,prec=100)),imaginary(numerical_approx(value,prec=100))
])
   j+=1
   if(j<502):
      x=random.random()
      if(x<.125):
         g(value^(1/8),array,j)
      elif((x>.125) and (x<.25)):
         g((sqrt(2)/2)*value^(1/8)+i*(sqrt(2)/2)*value^(1/8),array,j)
      elif((x>.25) and (x<.375)):
         g(i*value^(1/8),array,j)
      elif((x>.375) and (x<.5)):
         g((sqrt(2)/2)*-1*value^(1/8)+i*(sqrt(2)/2)*value^(1/8),array,j)
      elif((x>.5) and (x<.625)):
         g(-1*value^(1/8),array,j)
      elif((x>.625) and (x<.75)):
         g((sqrt(2)/2)*-1*value^(1/8)-i*(sqrt(2)/2)*value^(1/8),array,j)
      elif((x>.75) and (x<.875)):
         g(-i*value^(1/8),array,j)
      else:
         g((sqrt(2)/2)*value^(1/8)-i*(sqrt(2)/2)*value^(1/8),array,j)
   if(j>500):
      line(array,linestyle="dotted").show(figsize=[10,10])
array=[]
g(1,array,1)
```

We can recreate the inner-figure easily using parametric graphing techniques.



Sage Snippet:

```
p1=parametric_plot([cos(x),sin(x)],(x,0,2*pi),rgbcolor=hue(0.8))
p2=parametric_plot([-1/9*((7+1)*cos(x)-(cos((7+1)*x))),-1/9*((7+1)*sin(x)-
(sin((7+1)*x)))],(x,0,2*pi),rgbcolor=hue(0.6))
show(p1+p2,figsize=[8,8])
```

Do you notice the pattern?

The table below summarizes the important patterns that have emerged. The root function seems to always have one leaf less than the root being taken. For example, the cube root function will create a shape with just 2 leaves.

Clearly, the higher the root, the more leaves appear.

Root	2	3	4	5	6	7	8	9	∞
# of Leaves	1	2	3	4	5	6	7	8	∞
Shape Area	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{4\pi}{5}$	$\frac{5\pi}{6}$	$\frac{6\pi}{7}$	$\frac{7\pi}{8}$	$\frac{8\pi}{9}$	$\frac{9\pi}{10}$	1

It is also interesting to note that the most efficient way (least number of operations) way to generate these graphs is via randomness. A very chaotic picture morphs into a very orderly shape after enough repetitions.

Epiphany: Randomness must be incorporated into pure mathematics to preserve symmetry!

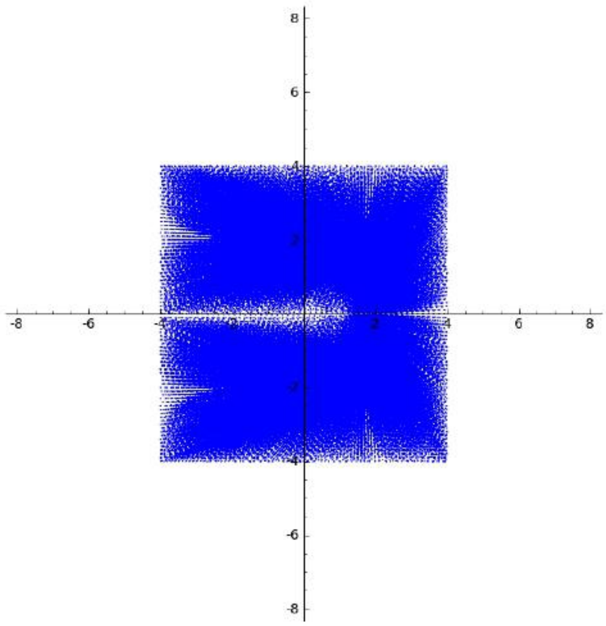
Roots over the entire complex domain

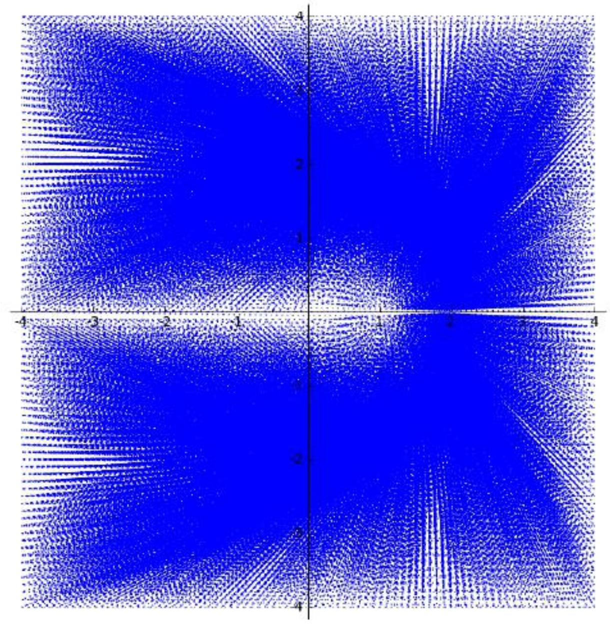
It is now time to get an eagle's eye view of this operation. Let's examine the *field* of the square root operation. To do so we will need to utilize a concept similar to a *vector field*. Without going into detail, we are going to draw a line from each point on the grid (complex domain) to its square root.

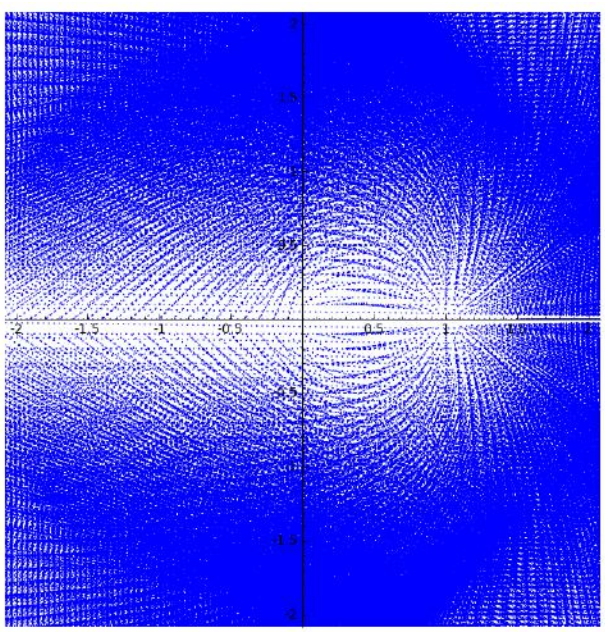
Our first example uses a 4X4 grid. Let us begin by first examining just the principal root.

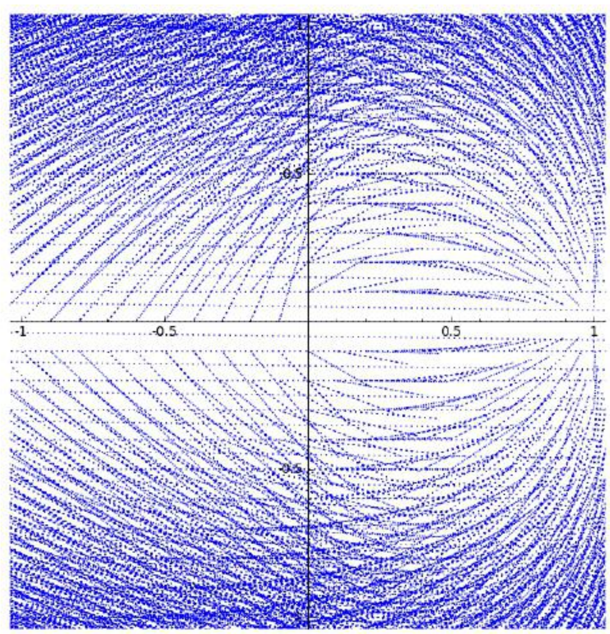
4X4 Box of Square Root (Just the principal root)

From the outside:

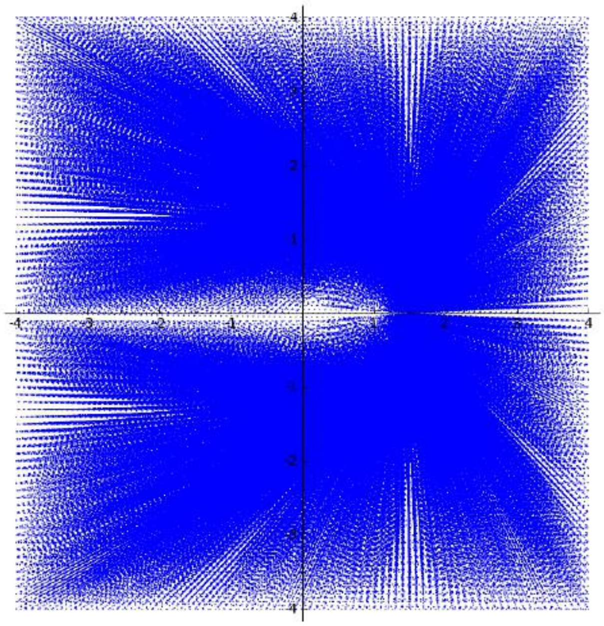




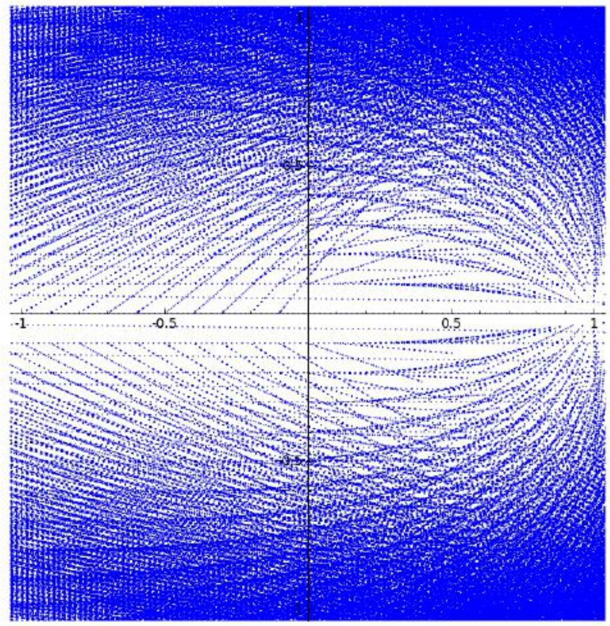




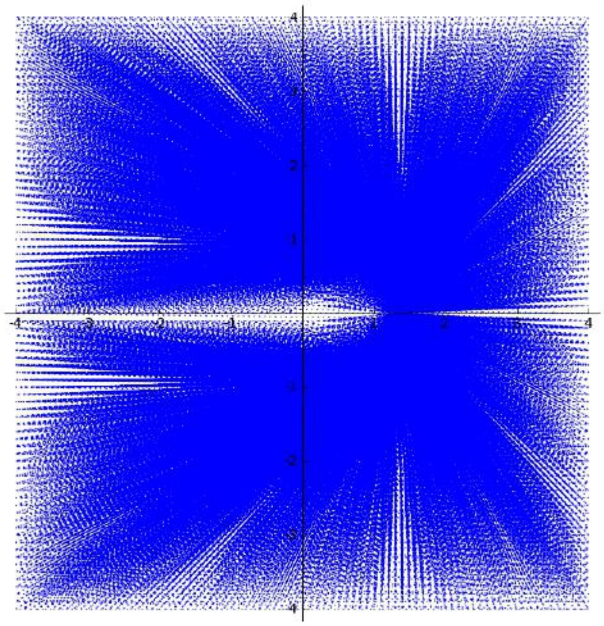
$X^{1/2}$



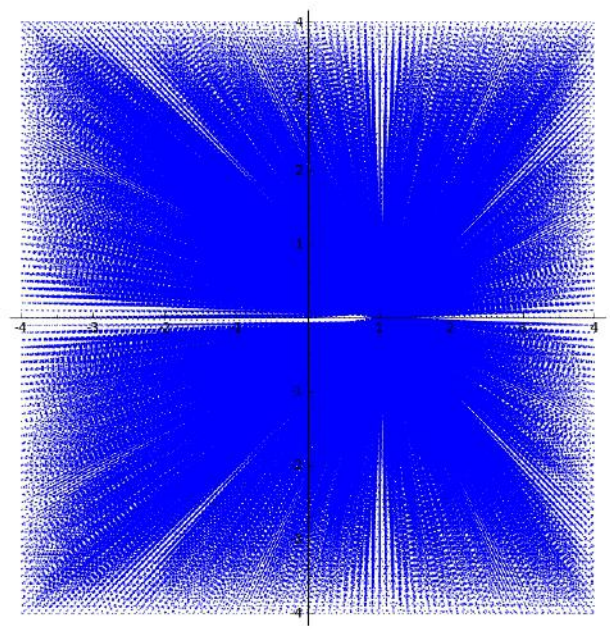
$X^{1/3}$



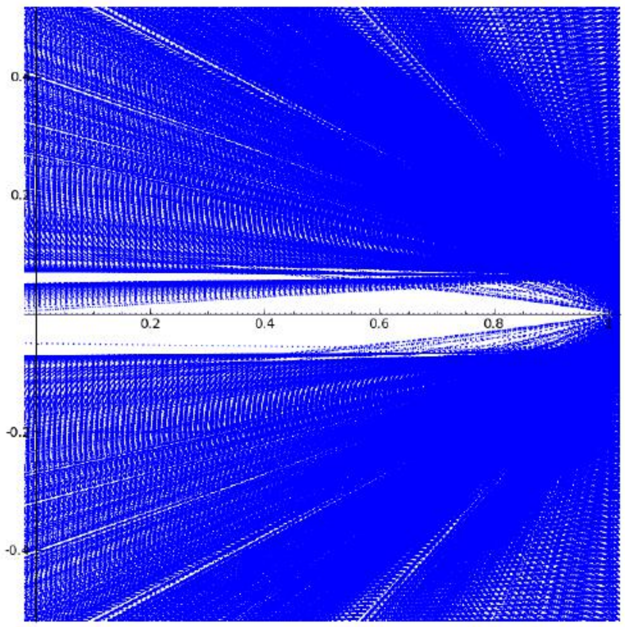
Zoomed In…



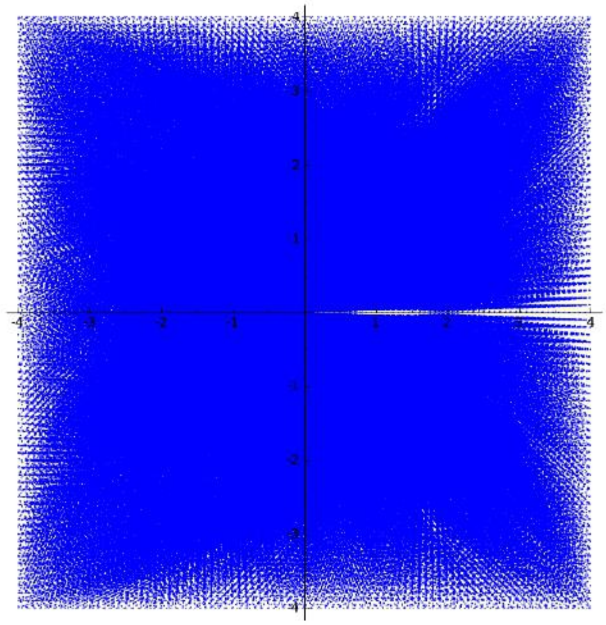
$X^{1/4}$



$X^{1/50}$



Zoomed in $\mathbf{X}^{1/50}$



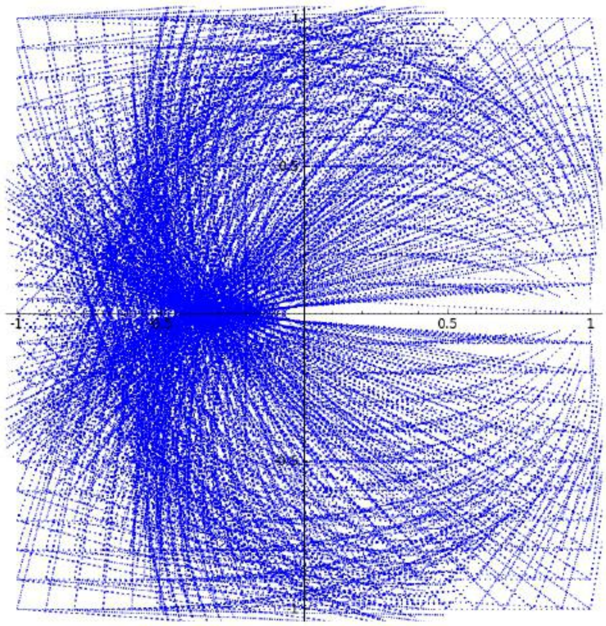
With Randomness...

Note these graphs because we will come back to them in the following chapter. It is interesting to note how positive 1 appears to be the center of activity. We will be comparing these graphs to their counter-parts: X^2, X^3, X^4, …

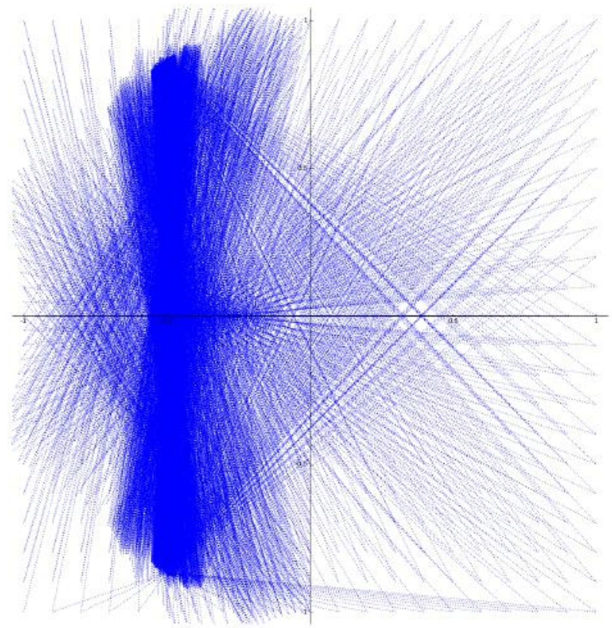
For the sake of completeness, we will examine the non-principal square root.

Non-Principal Square Roots (1X1 Grid Analysis):

Single Iteration (non-principal square root)



Many Iterations (non-principal square root)



While these graphs may look interesting, our real goal will be to combine the principal and non-principal root as we did before.

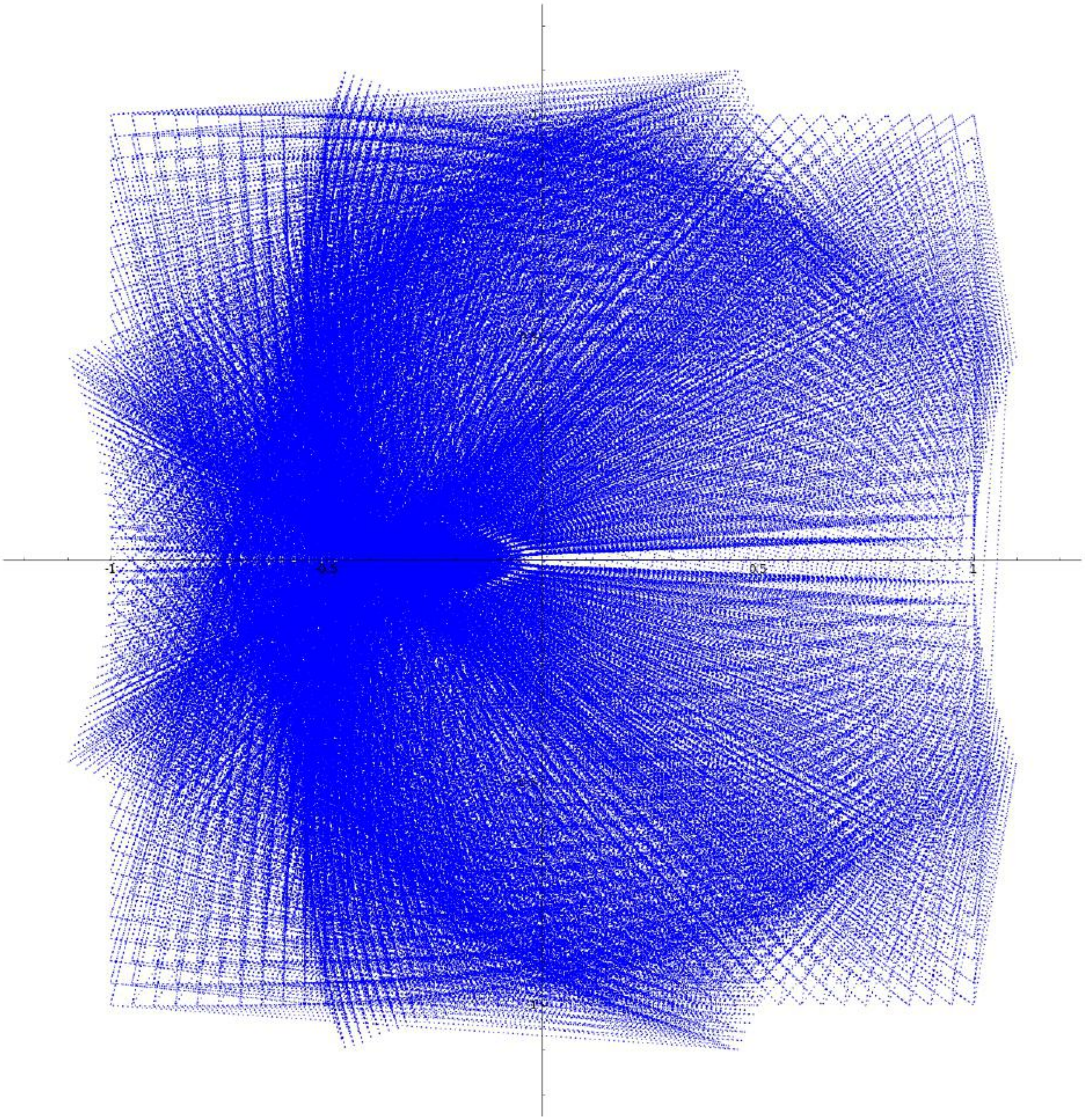
Sage Snippet:

```
#Squaring Experiment
import random
def g(value,array,i,q):
   #print «value=»,value
   array.append([real(numerical_approx(value,prec=100)),imaginary(numerical_approx(value,prec=100))])
   array.append([real(numerical_approx(value^(1/2),prec=100)),imaginary(numerical_approx(value^(1/2),prec=100))])
   array.append([real(numerical_approx(value,prec=100)),imaginary(numerical_approx(value,prec=100))])
   array.append([real(numerical_approx(-1*value^(1/2),prec=10)),imaginary(numerical_approx(-1*value^(1/2),prec=10))])
   array.append([real(numerical_approx(value,prec=100)),imaginary(numerical_approx(value,prec=100))])
   i+=1
   if(i<5):
      #x=random.random()
      #if(x<.5):
      g(-1*value^(1/2),array,i,q)
      #else:
      g(1*value^(1/2),array,i,q)
```

```
array=[]
m=0
while(m<21):
    p=0
    while(p<21):
        #print "m=",m
        g(-1+(.1*m)+(.1*p*1*sqrt(-1))-1*1*sqrt(-1),array,1,1)
        #g(-1+(.1*m)+(.1*p*1*sqrt(-1))-1*1*sqrt(-1),array,1,2)
        p+=1
    m+=1

line(array,linestyle="dotted").show(xmin=-1.2, xmax=1.2, ymin=-1.2, ymax=1.2,figsize=[16,16])
print "done"
```

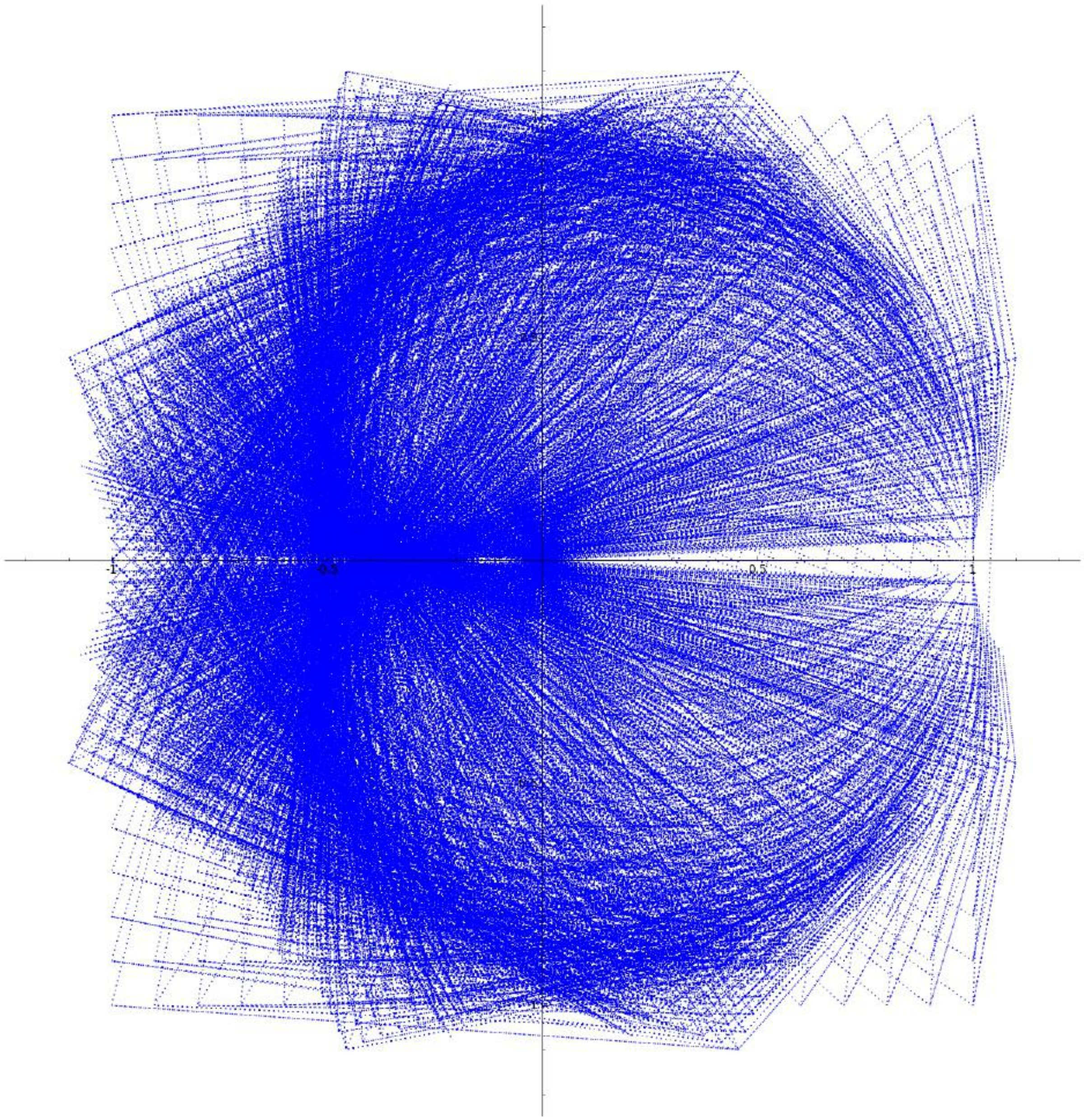
1X1 (Single Iteration, principal and non-principal, always one way)

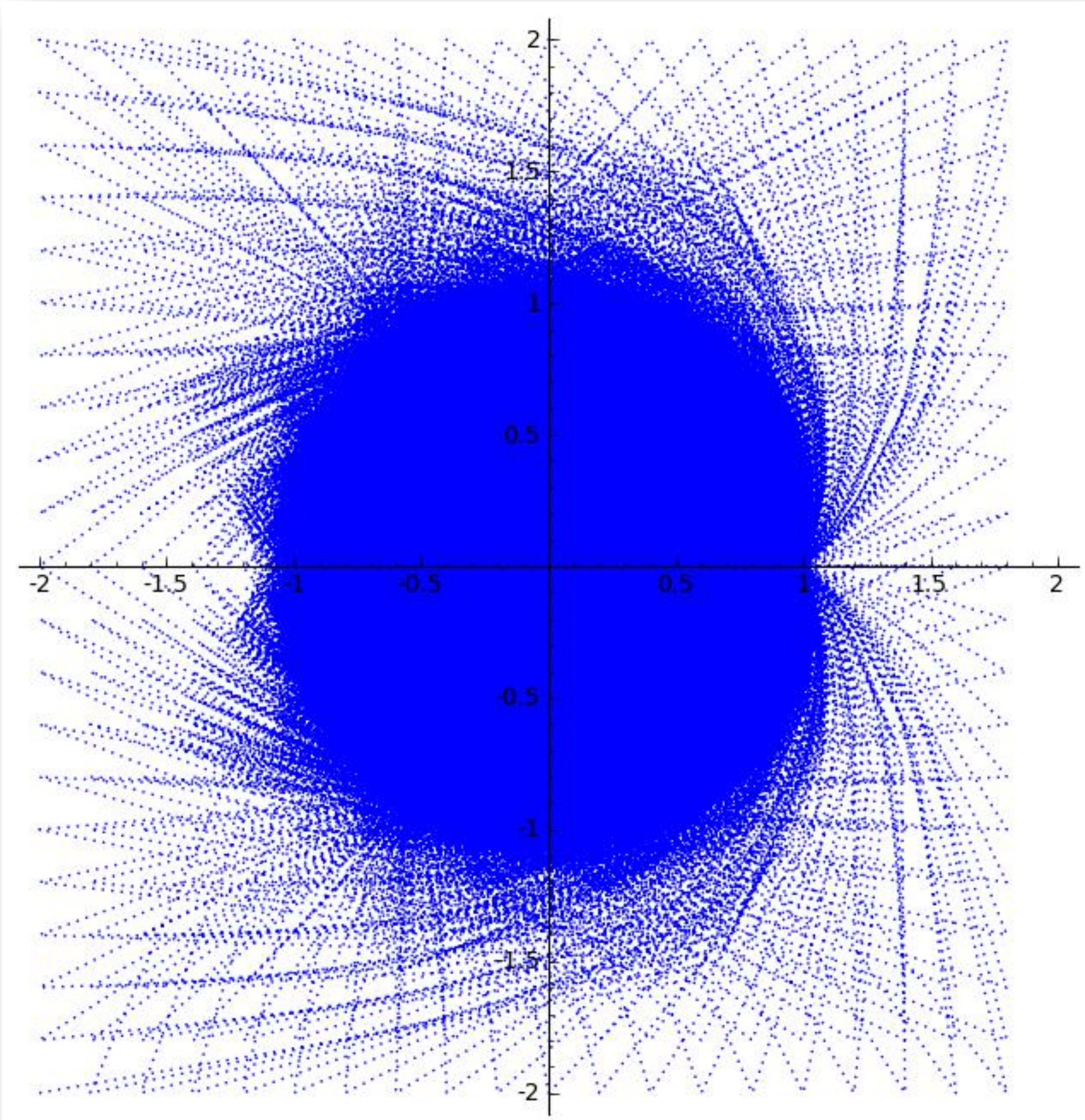


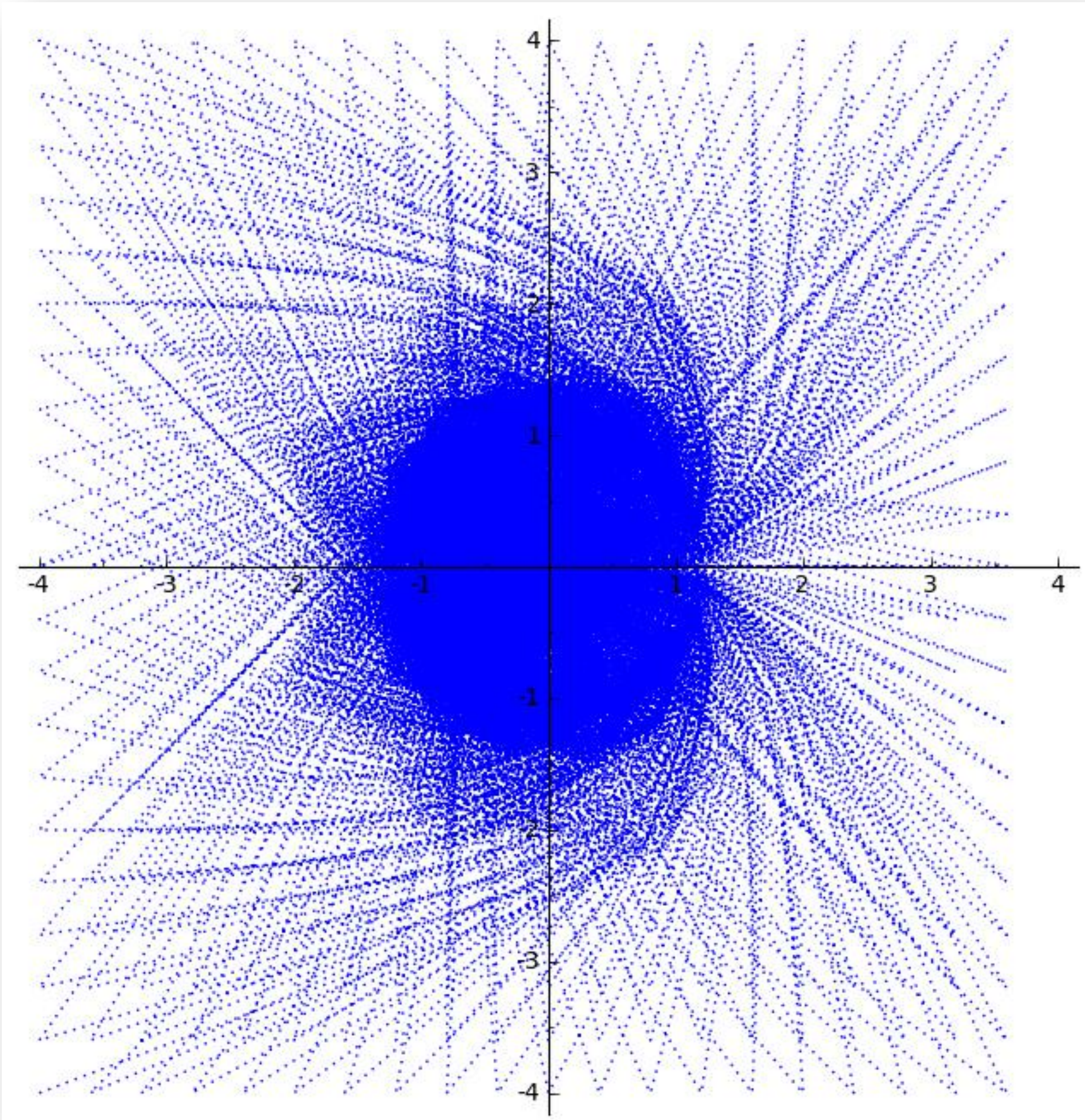
Discovery: each point outside the 1X1 box is $\sqrt{\sqrt{2}}$ in length

1X1 (Two Iterations, principal and non-principal, always one way)

You can see the object slowly taking shape…

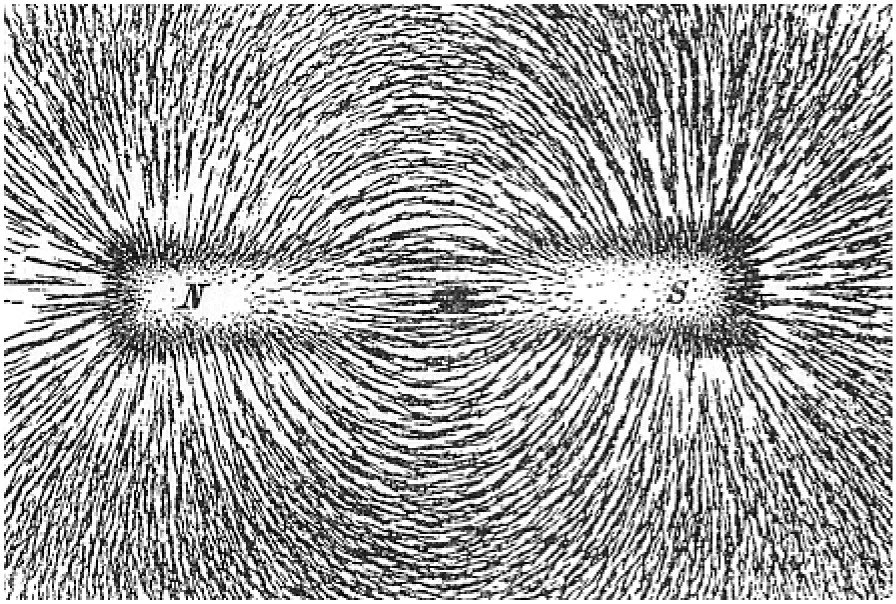



2
1.5
1
0.5
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0.5
1
1.5
2


4
3
2
1
-1
-2
-3
-4
-4
-3
-2
-1
1
2
3
4

Reminiscent of Gravitational and Electrical Fields

The previous graph appears to have similar characteristics to a magnetic field shown as illustrated below.



Chapter 3: Infinite Powers

Introduction

This chapter has similar goals to our last chapter. Rather than studying fractional powers (e.g. square roots, cube roots, etc.), we will instead be examining whole number exponents in this chapter. In particular, we will be looking at the complex plane and how it behaves. Additionally, we should compare these functions with their inverse, which we just studied in the previous chapter. Our first question will be how does the square root behave compared to the squaring operation?

Our First Power Experiment

To examine the *field* that is generated by performing the exponent operation, we must examine how this *field* affects a set of points. We are going to set up points in a few different ways:

- A few sets of **horizontal** points.
- A few sets of **vertical** points.
- A **grid of points** that form a box spaced evenly from each other.

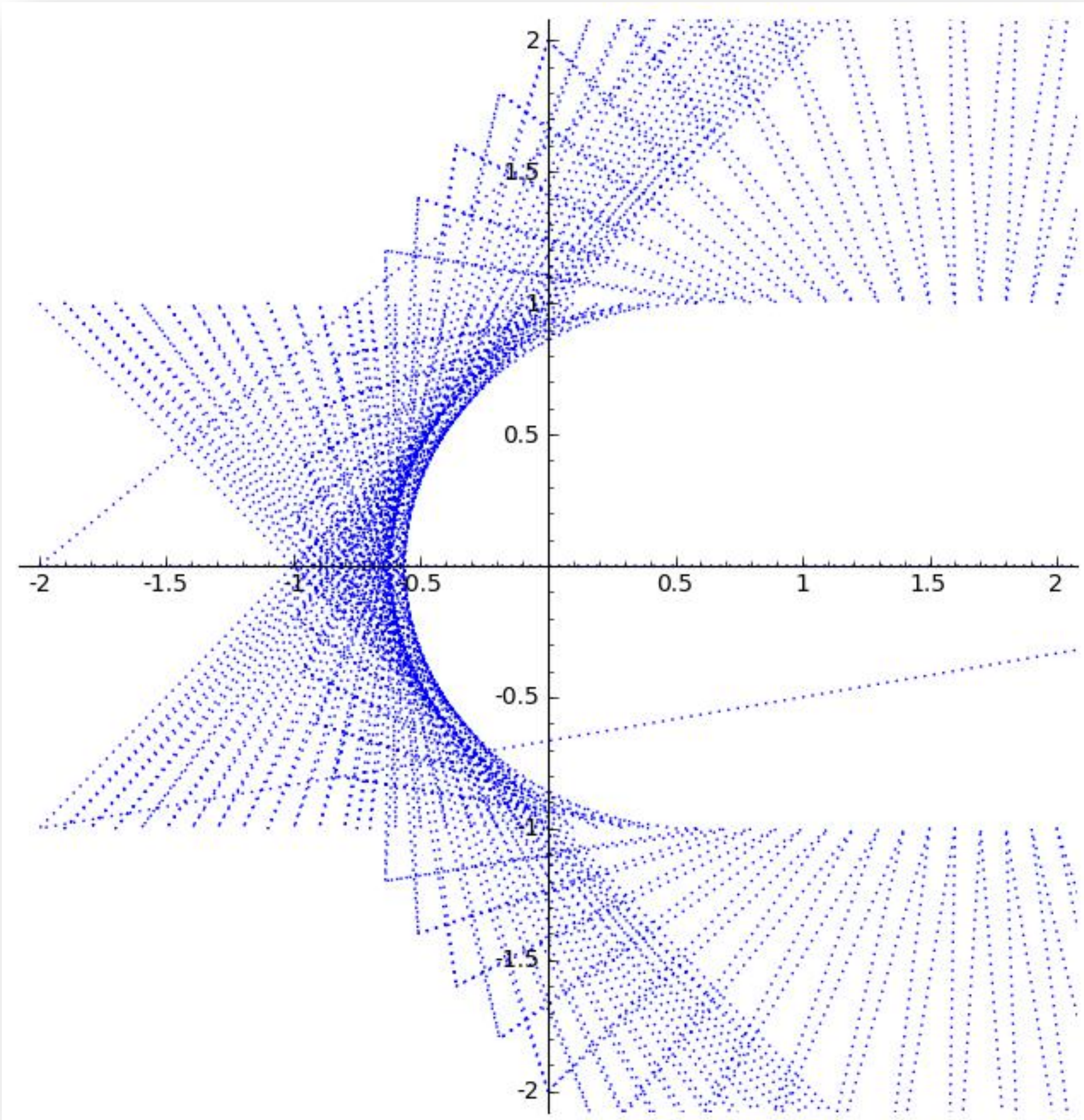
This will help us get the full picture of how this field works. The reason we must examine this a few different ways is that there is so much information that our pictures could easily be overloaded with data and we will miss the underlying behavior. We must examine both large and small scale when looking at these graphs.

The code below shows how we graph our original points. Then these points create a *field* by multiplying each one by itself. We have chosen our selection for spacing of these points to be .1 units between each other horizontally. We will do this 3 times. There will be a separate horizontal line at *i*, 0, and –*i*.

Sage Snippet:

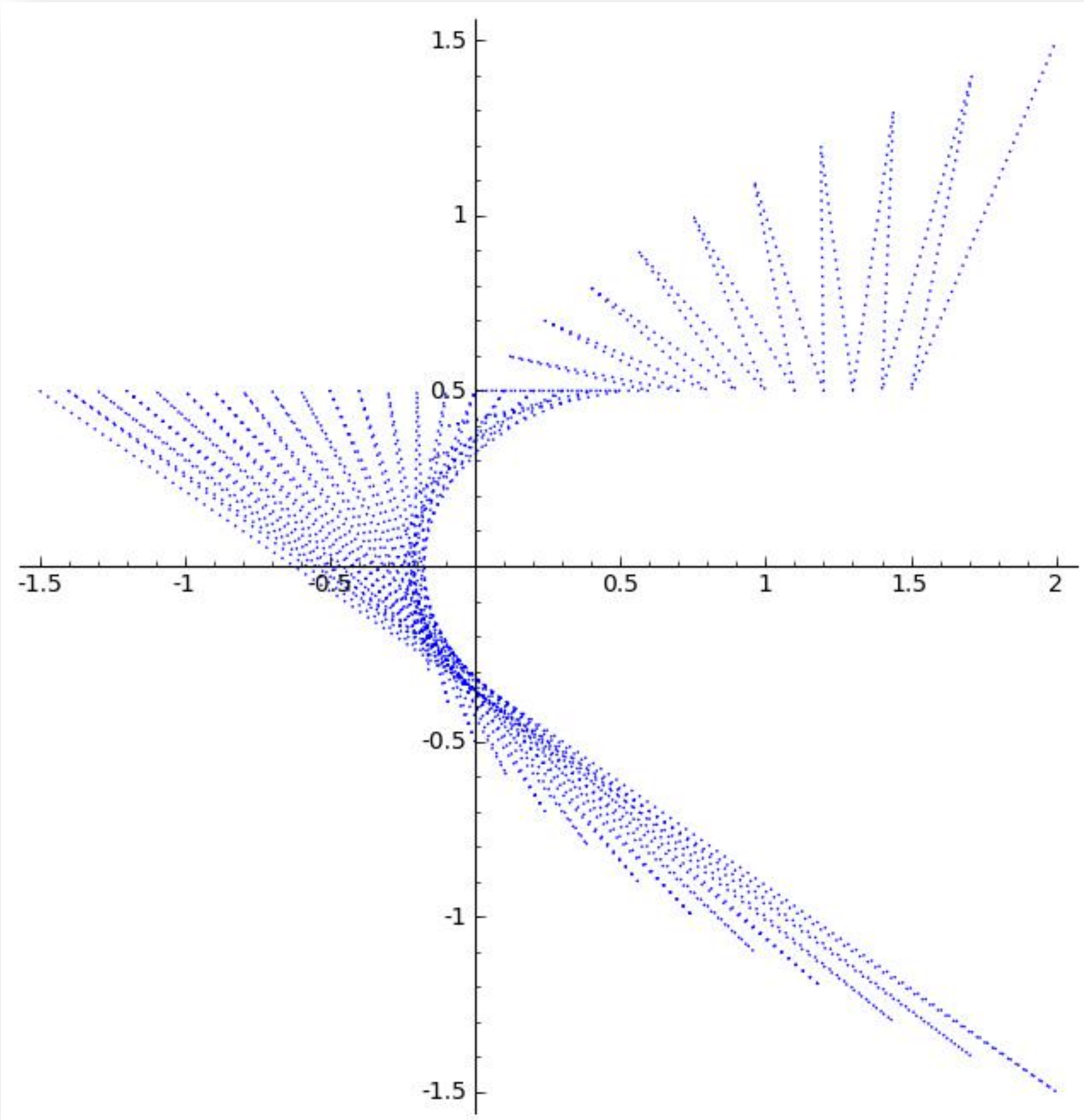
```
#Squaring Experiment
import random
def g(value,array,i)
   array.append([real(numerical_approx(value,prec=100)),imaginary(numerical_approx(value,prec=100))
]) array.append([real(numerical_approx(value^2,prec=100)),imaginary(numerical_approx(value^2,prec=
100))])
   i+=1
   if(i<42):
      g(value+.1+0*.1*sqrt(-1),array,i)
array=[]
p=0
while(p<8):
   g(-2-.5*p*sqrt(-1)+2*sqrt(-1),array,1)
   p+=1
line(array,linestyle="dotted").show(xmin=-4, xmax=4, ymin=-4, ymax=4,figsize=[8,8])
```

X^2 (Horizontal Line at i,0,and –i of length 2)



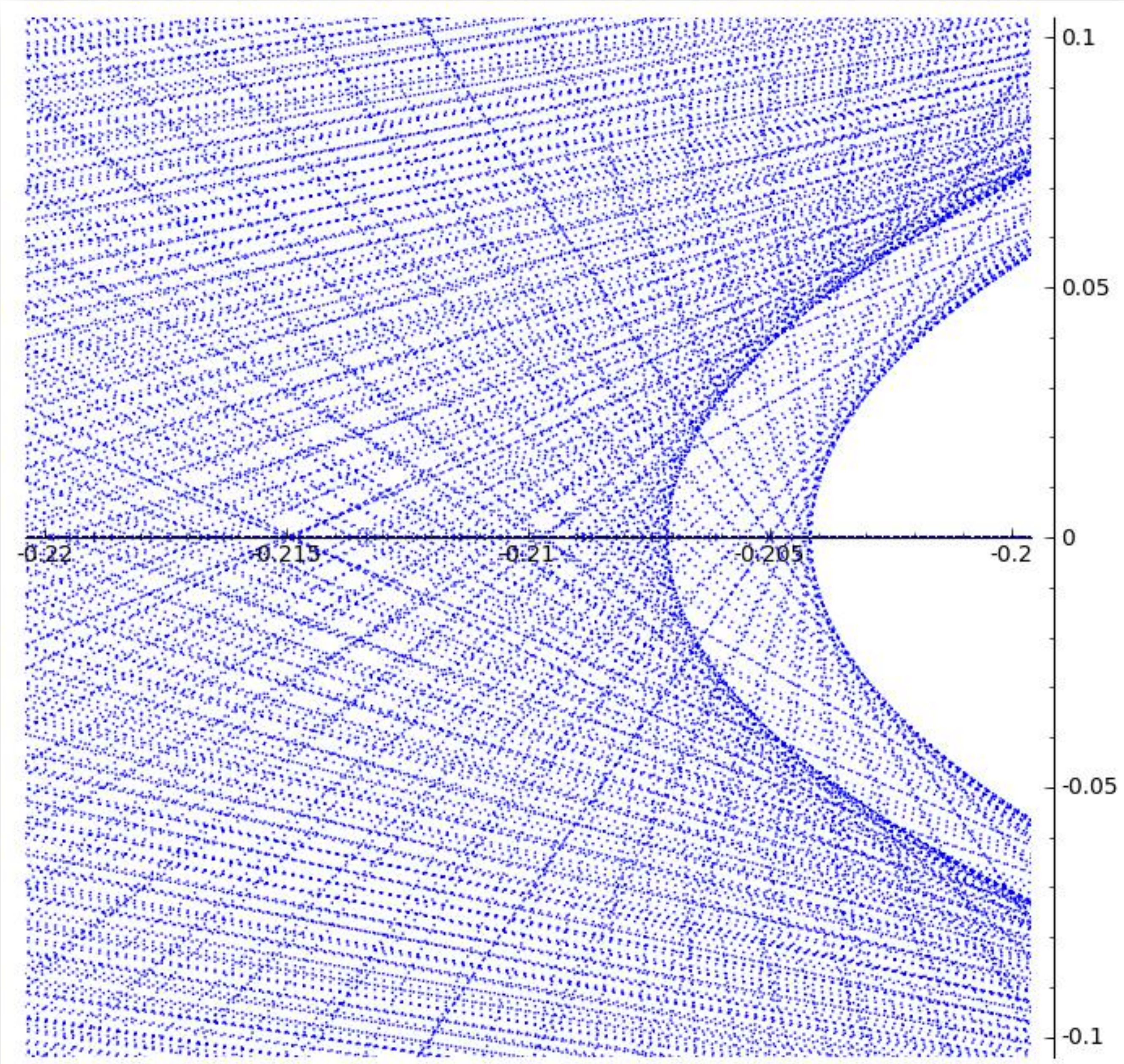
The results seem to have some interesting behavior near the real axis.

X^2 (Single Horizontal Line starting at .5i)



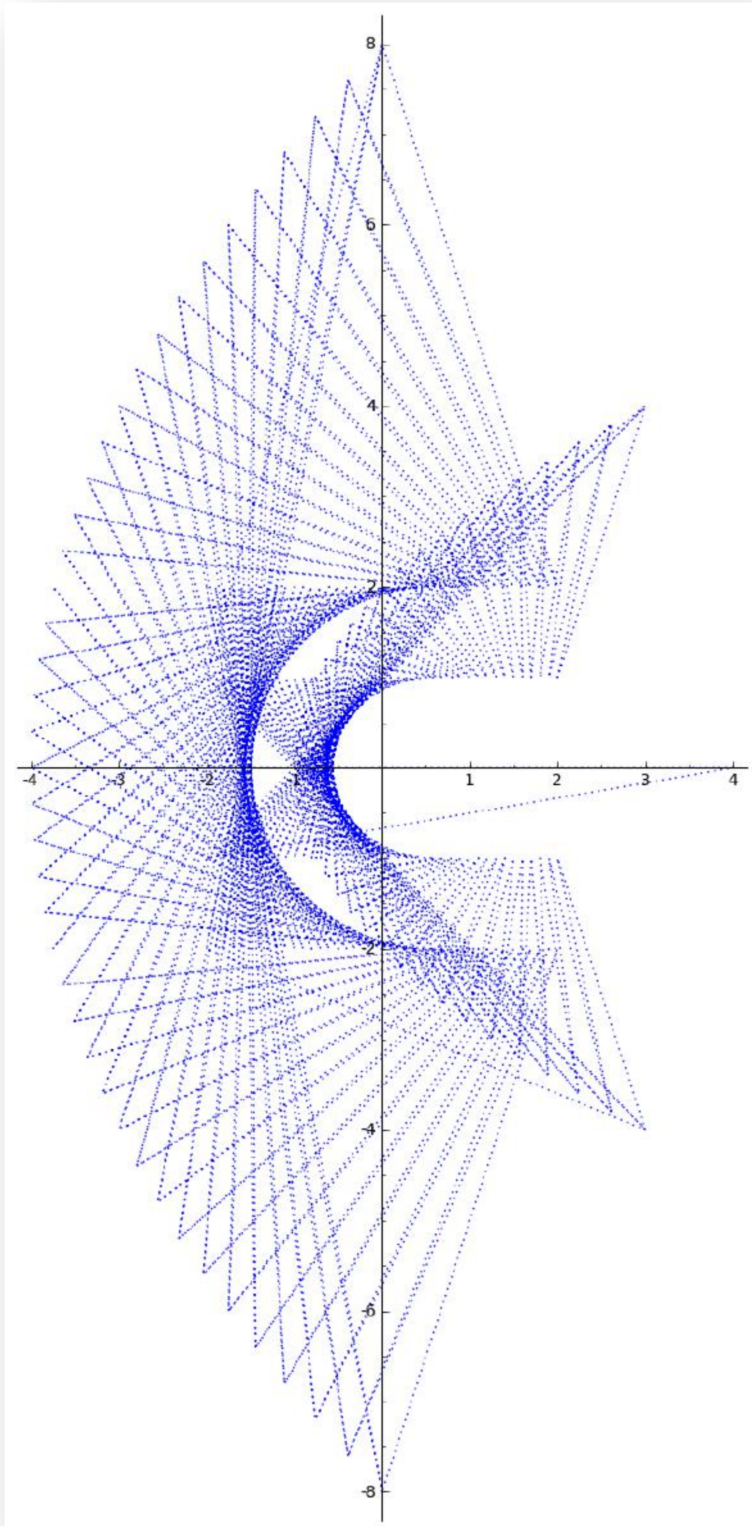
We see almost exactly the same behavior when we draw the horizontal line at ½ i instead of i.

Zooming in on the X axis...



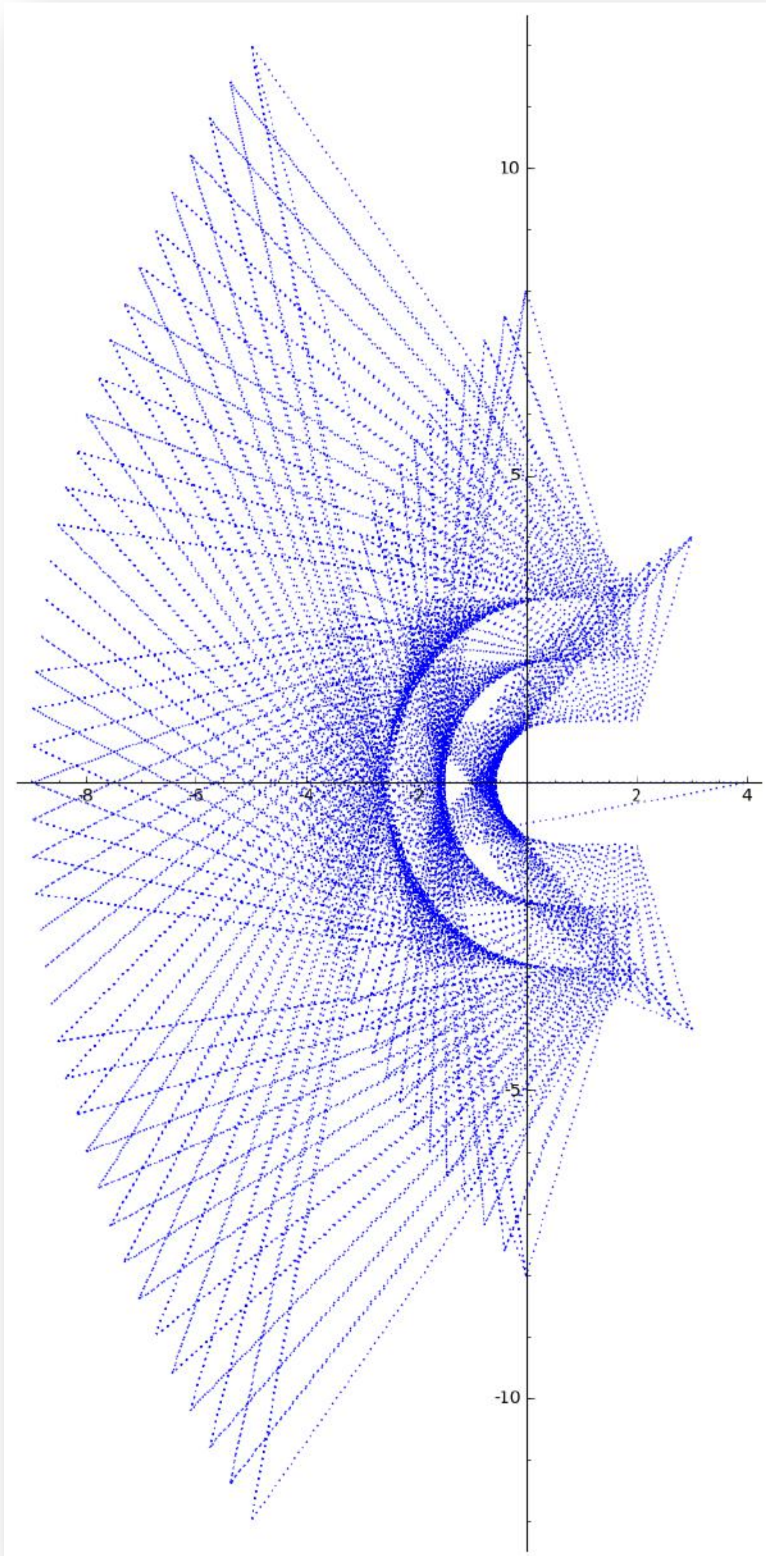
If we look at the real axis very closely, we can see there is actually a disk forming. We see this behavior no matter where we draw the horizontal line.

X^2 on a horizontal line at 2,1,0,-1,-2



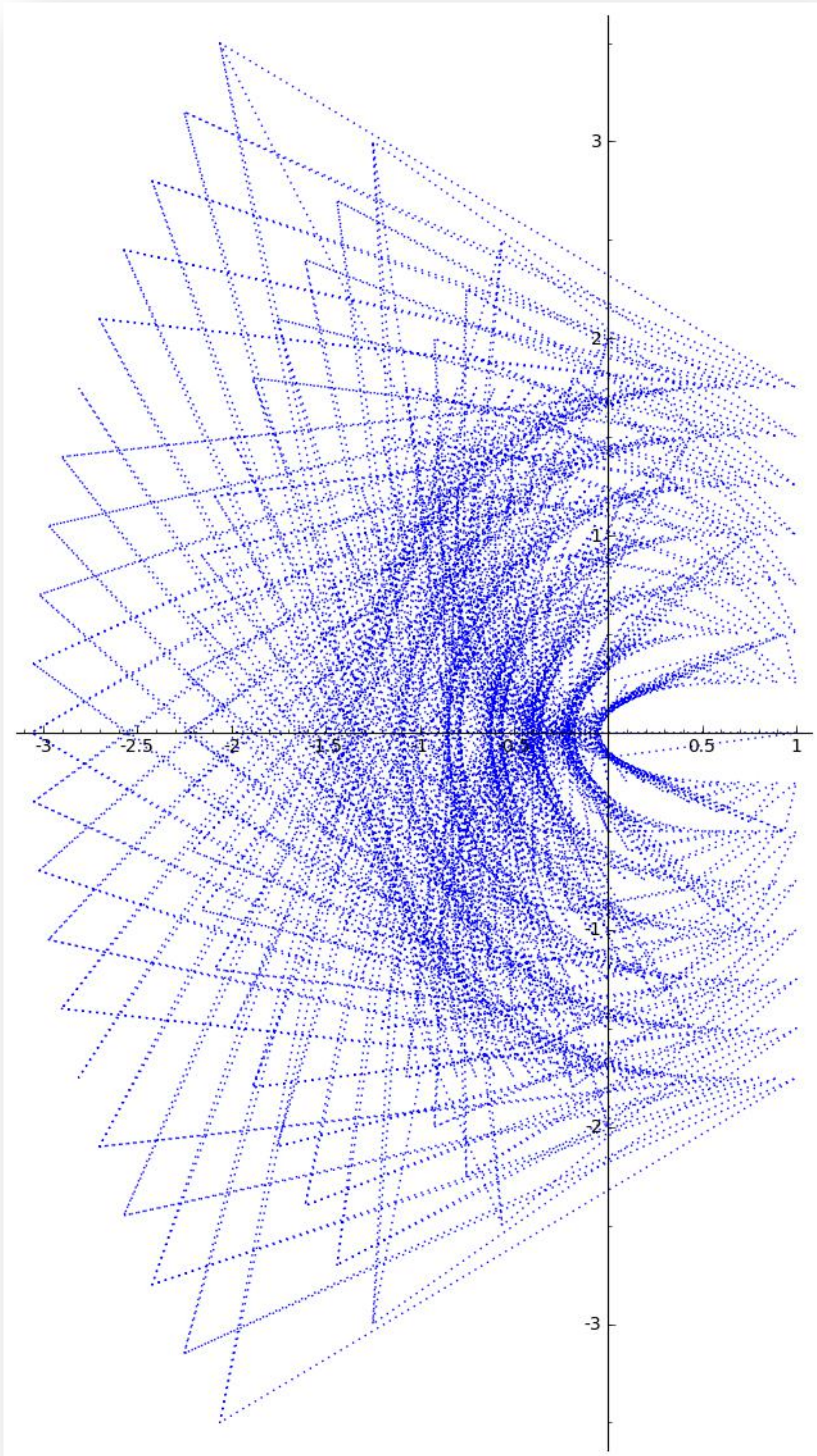
You will begin to notice that the overall height appears to be twice that of the width.

X^2 on a horizontal line at 3,2,1,0,-1,-2,-3

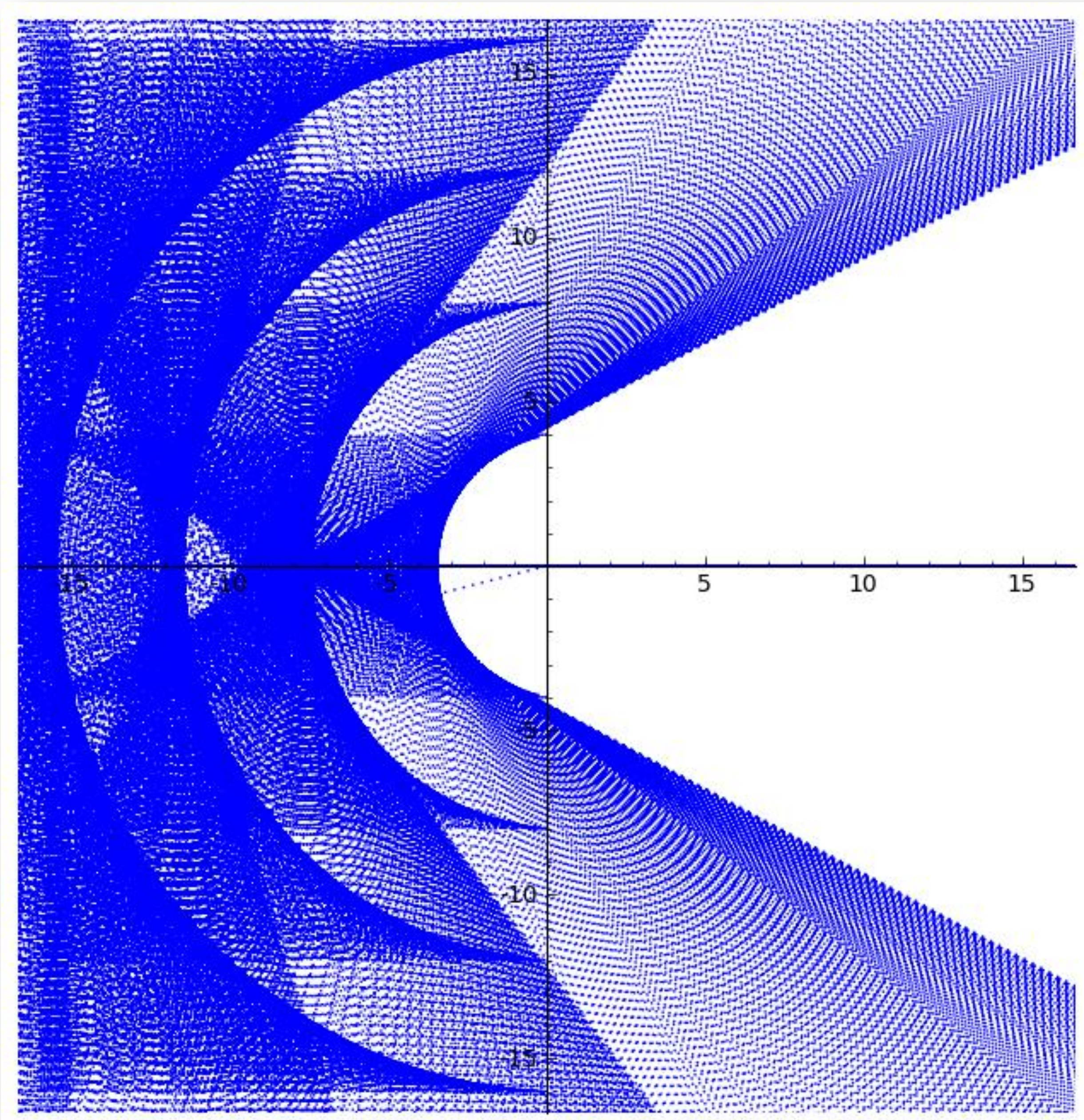


Every horizontal field line we add appears to create another semi-ellipse on the inside of the object.

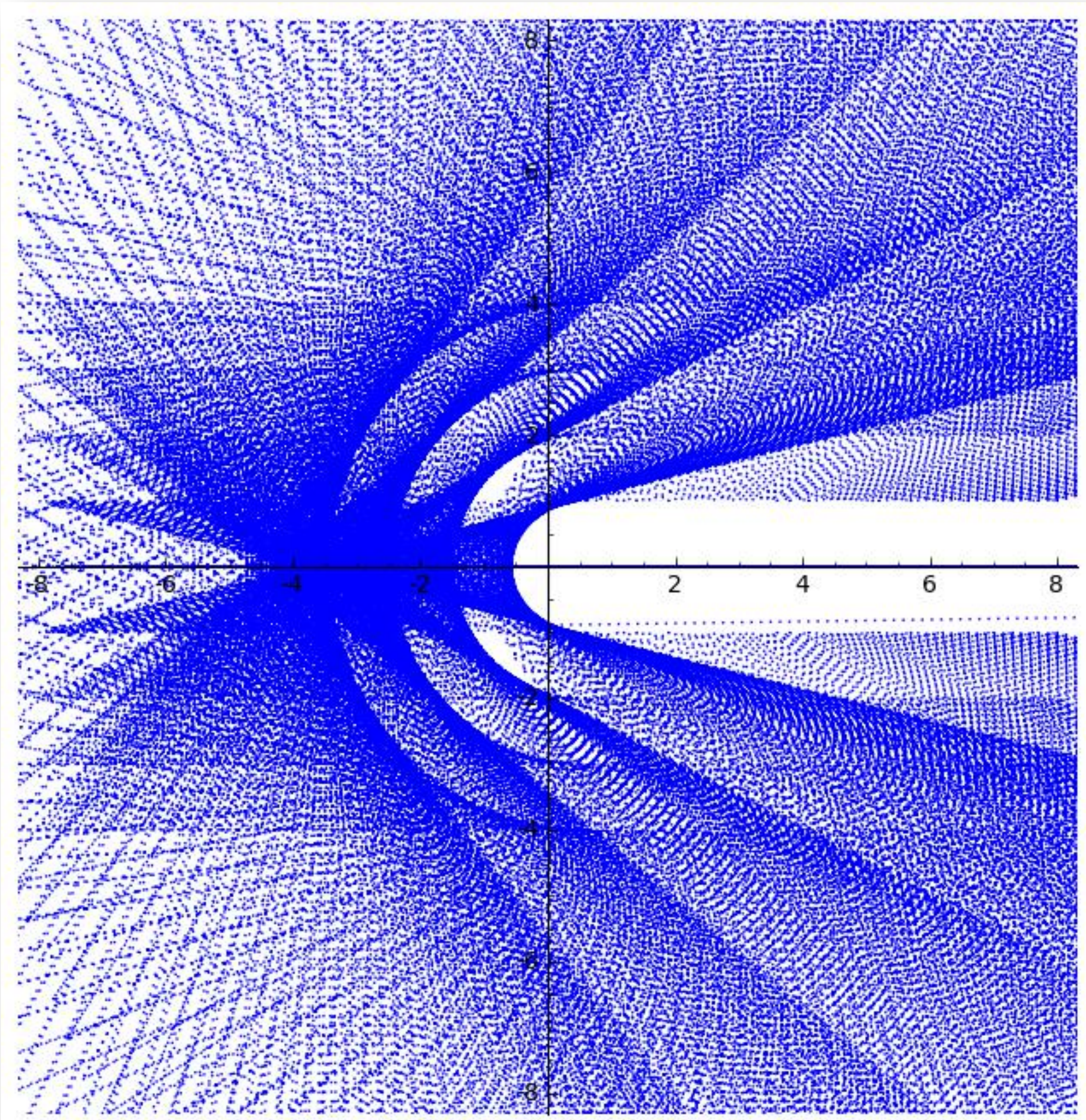
We repeat this experiment at lower fidelity, which means there are fewer points used in each horizontal line. However, in total, we are adding more horizontal lines to see the field. It seems that there is a fairly conclusive pattern forming.


3
2
1
-1
-2
-3
-3
-2.5
-2
-1.5
-1
-0.5
0.5
1

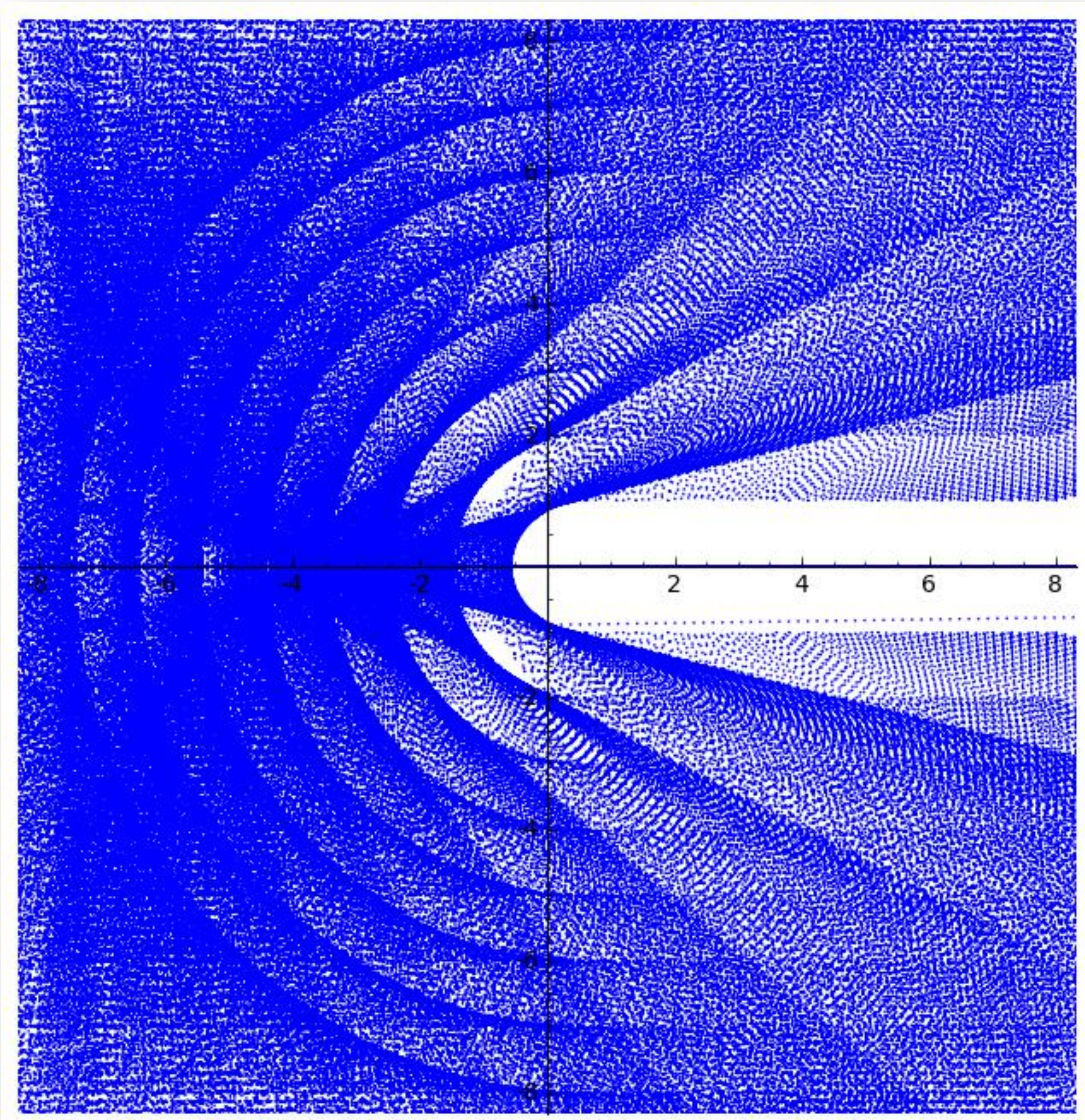
With 8 iterations...



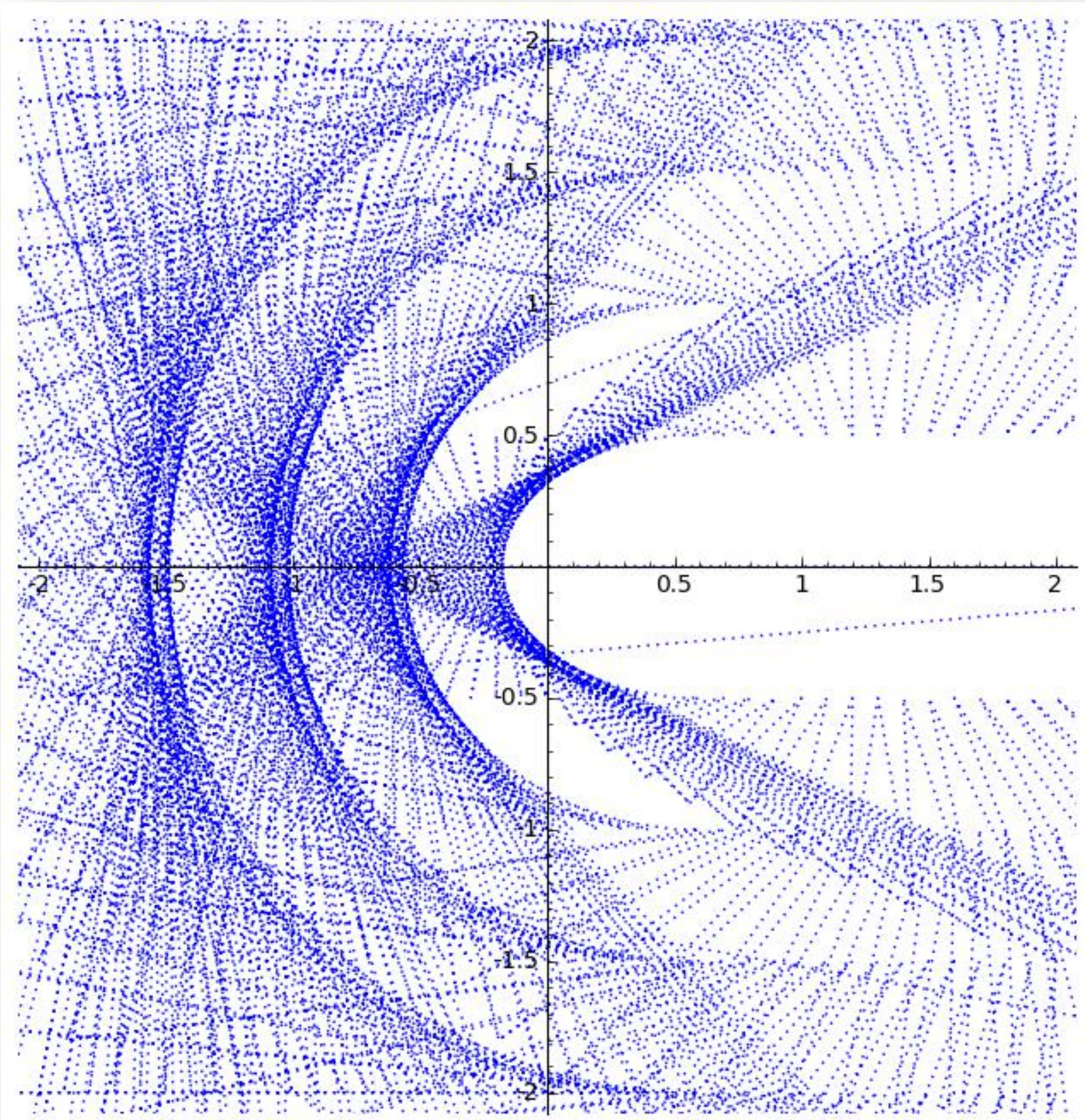
With more iterations…



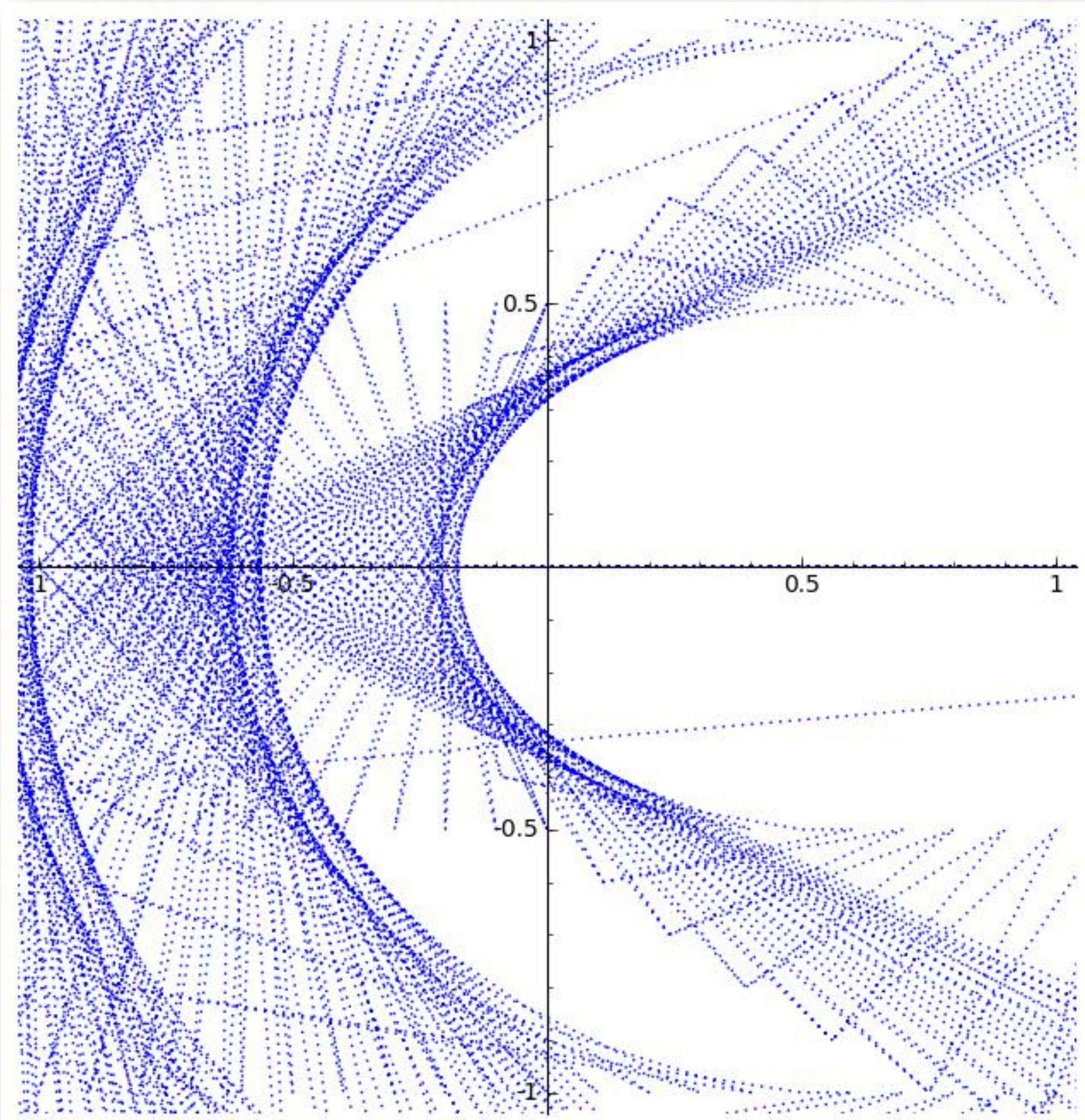
With even more iterations…



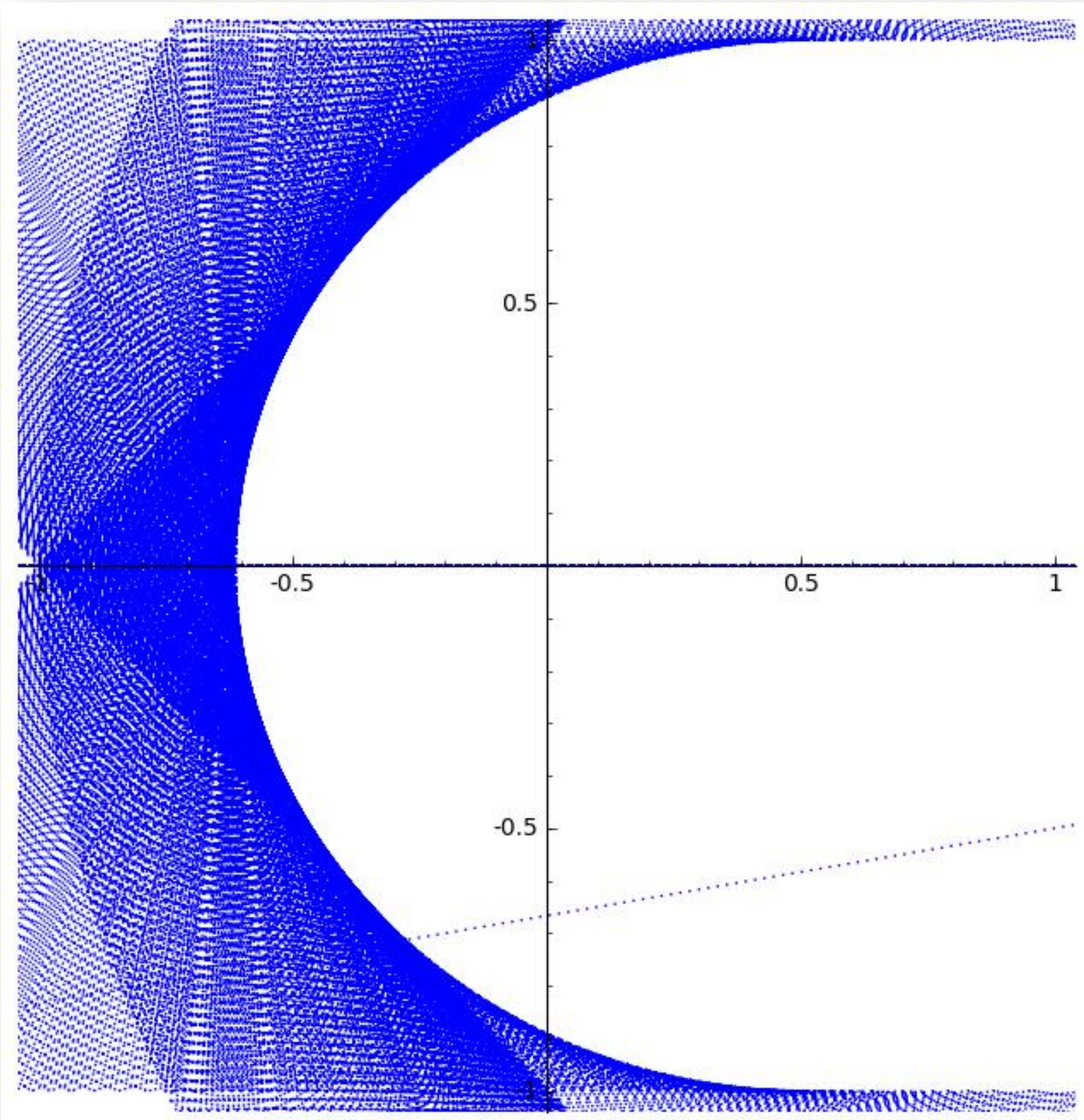
Zooming in...



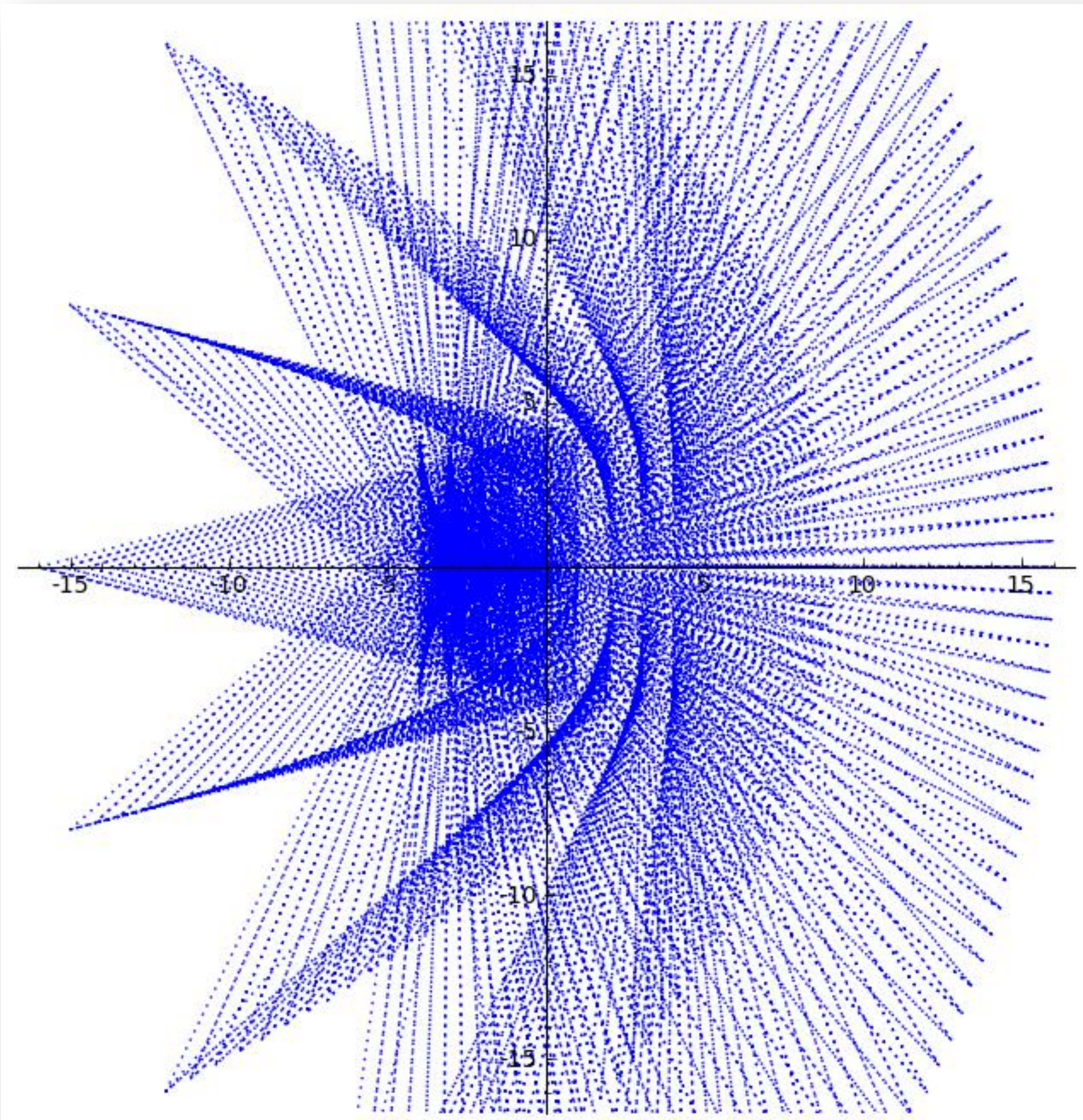
X^2 (Zooming in about the origin)



Did you notice the first arch we created on the horizontal line drawn at 1 imaginary unit? Can you guess the intersection point? If not, we come back to this soon. Hint: it's a ratio and some people say it is *golden.*



Now, what if we draw our field lines vertically rather than horizontally.



You will notice here the vertical axis is much larger than the horizontal axis by a factor of 2. Perhaps when we combine both the vertical and horizontal fields, we should expect to see a figure much taller than wide.

Creating Field Lines Starting with a Grid

Rather than biasing our results with graphs in the horizontal or vertical direction, let's instead start from a simple grid and see how these results compare to our previous work.

Because taking any number to a power creates very large results very quickly, we will need to zoom out considerably to see the full picture. We also will be zooming into the center to look for any patterns. This is computationally very intensive so we will be selective on how we create our grids.

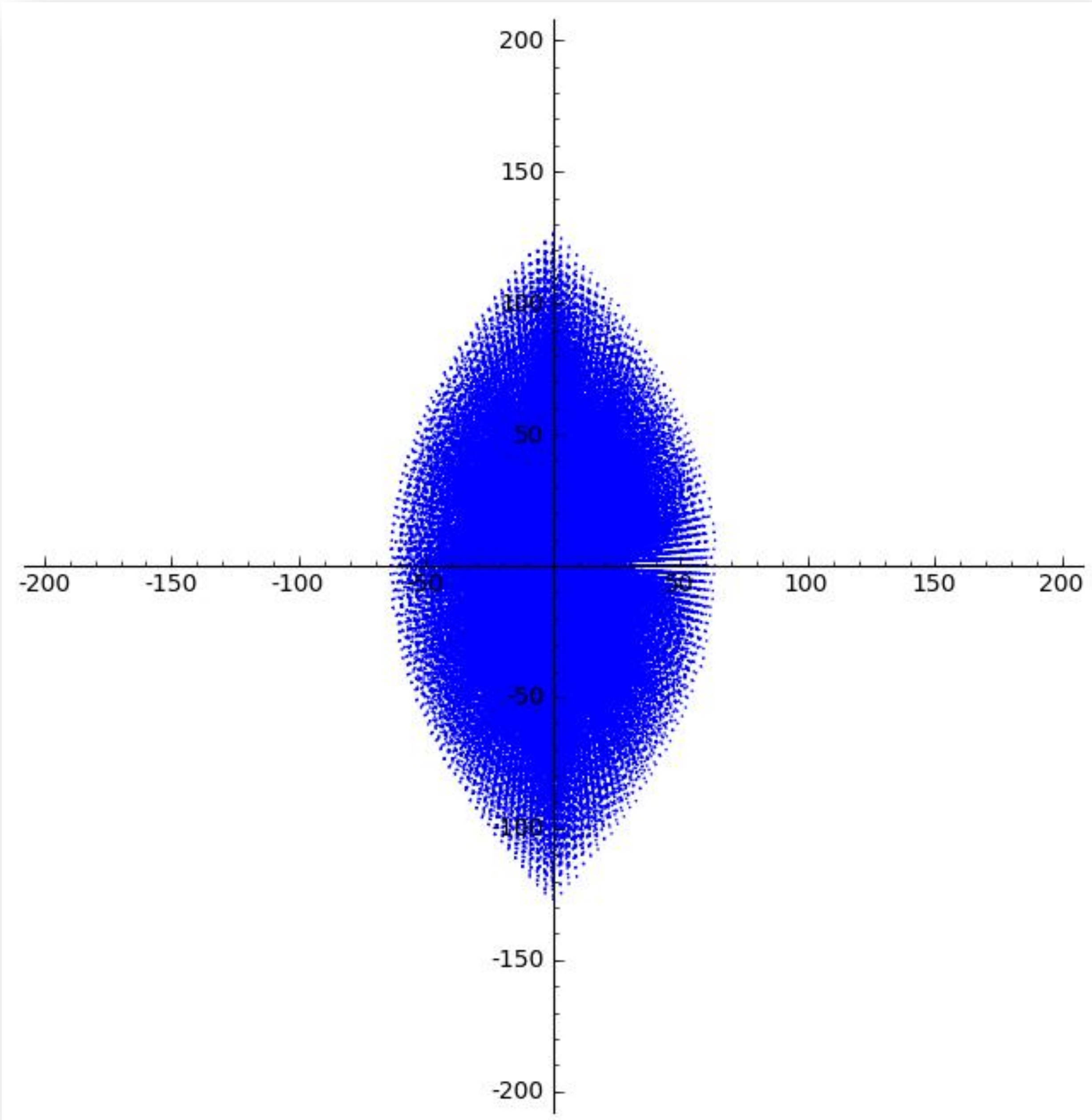
We will begin by creating a grid of points that form a box: 8 units by 8 units. You will see a cat's eye formation. However, you will see a slight asymmetry on the positive horizontal axis about the origin.

The following code provides a general outline on how we can iterate through a set of points and produce results graphically. The below example is for a 4X4 grid but could easily be modified for an 8X8 grid.

Sage Snippet:

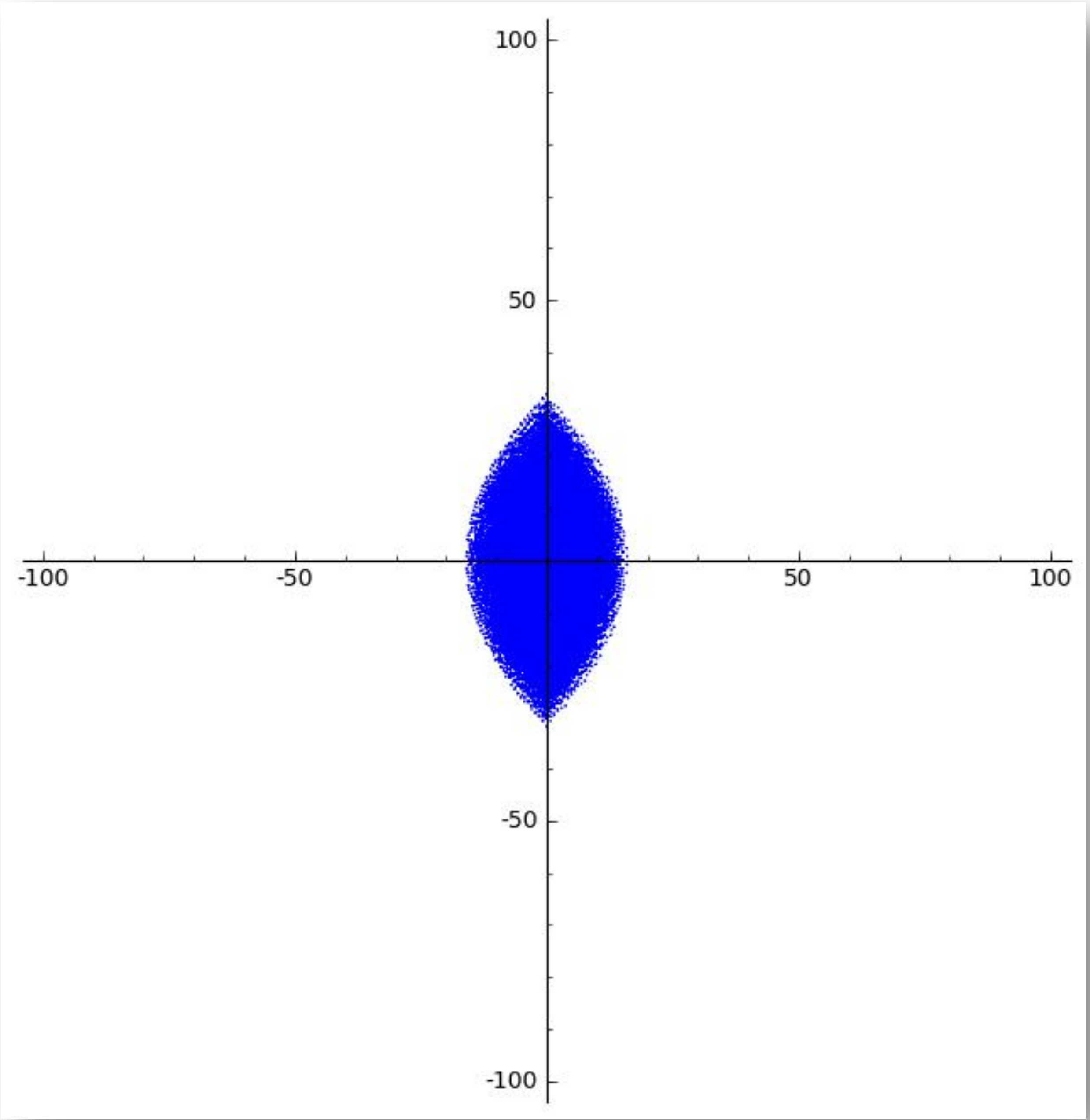
```
#Squaring Experiment 4X4 Grid
import random
def g(value,array,i):
   array.append([real(numerical_approx(value,prec=100)),imaginary(numerical_approx(value,prec=100))
])
array.append([real(numerical_approx(value^2,prec=100)),imaginary(numerical_approx(value^5,prec=10
0))])
array.append([real(numerical_approx(value,prec=100)),imaginary(numerical_approx(value,prec=100))])
   i+=1
   if(i<42):
      g(value+.0+1*.2*sqrt(-1),array,i)
array=[]
p=0
while(p<40):
   g(-4+.2*p+0*sqrt(-1)-4*1*sqrt(-1),array,1)
   p+=1
line(array,linestyle="dotted").show(xmin=-4, xmax=4, ymin=-4, ymax=4,figsize=[8,8])
```

X^2 - Plotted from a grid of 8X8



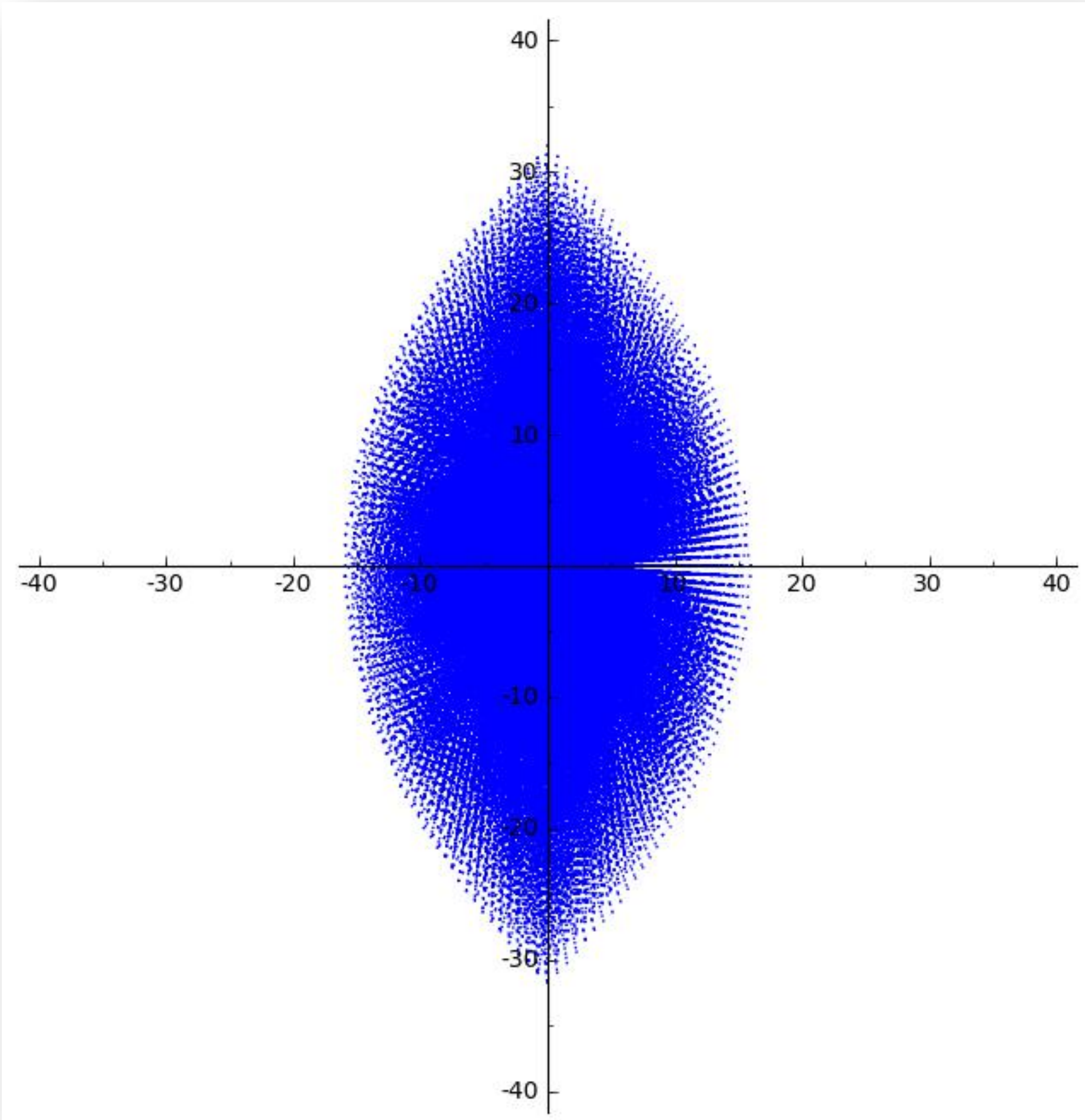
This graph combines the behavior of our previous graphs into a very orderly bigger picture. We can see how X^2 looks in the imaginary world. However, this is just from the exterior.

X^2 - Plotted from a grid of 4X4

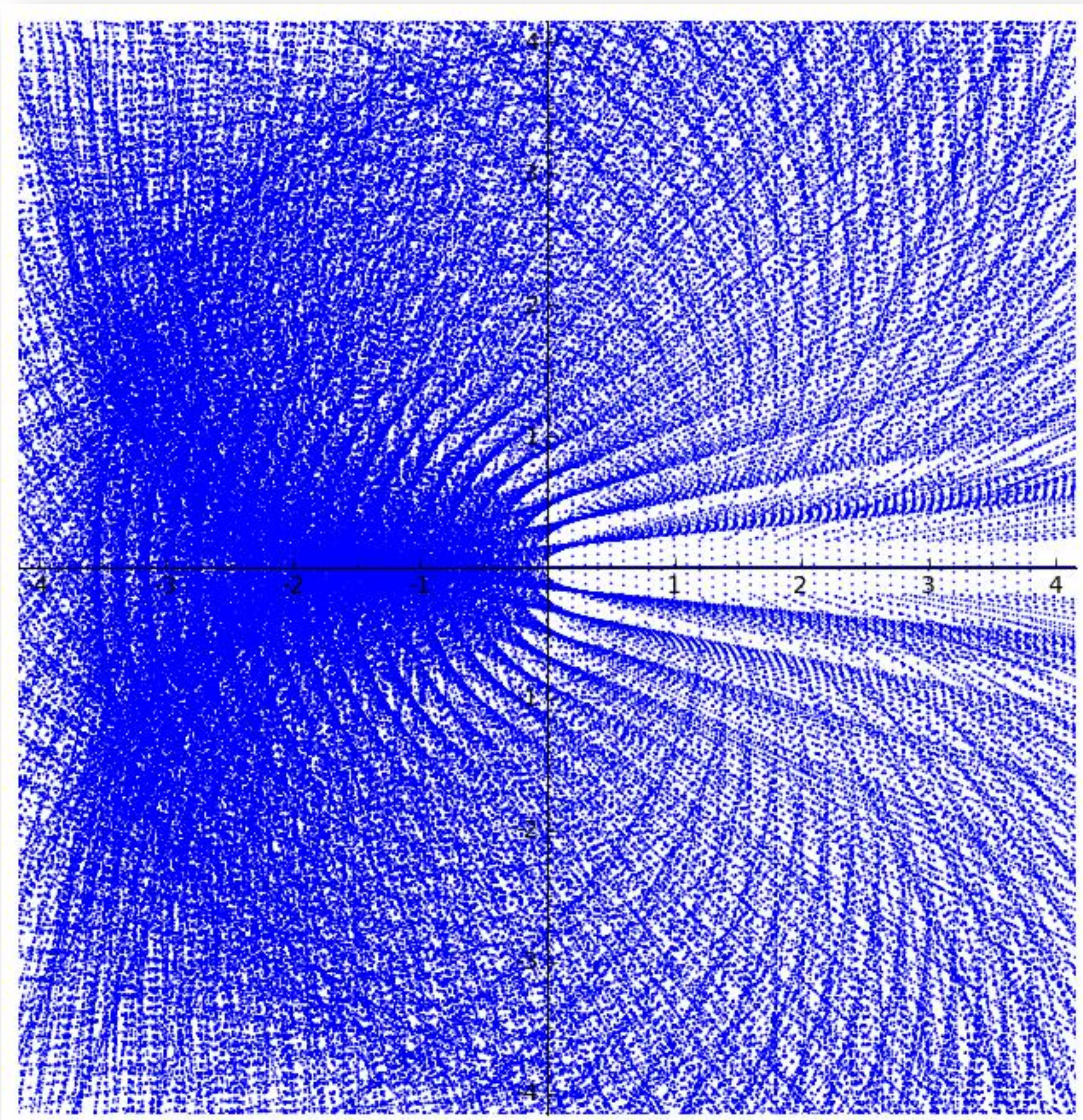


Even with a simple 4X4 grid of points. The X^2 operation is so powerful we have to zoom out about 50 units to view it properly.

Zooming in to the picture above:

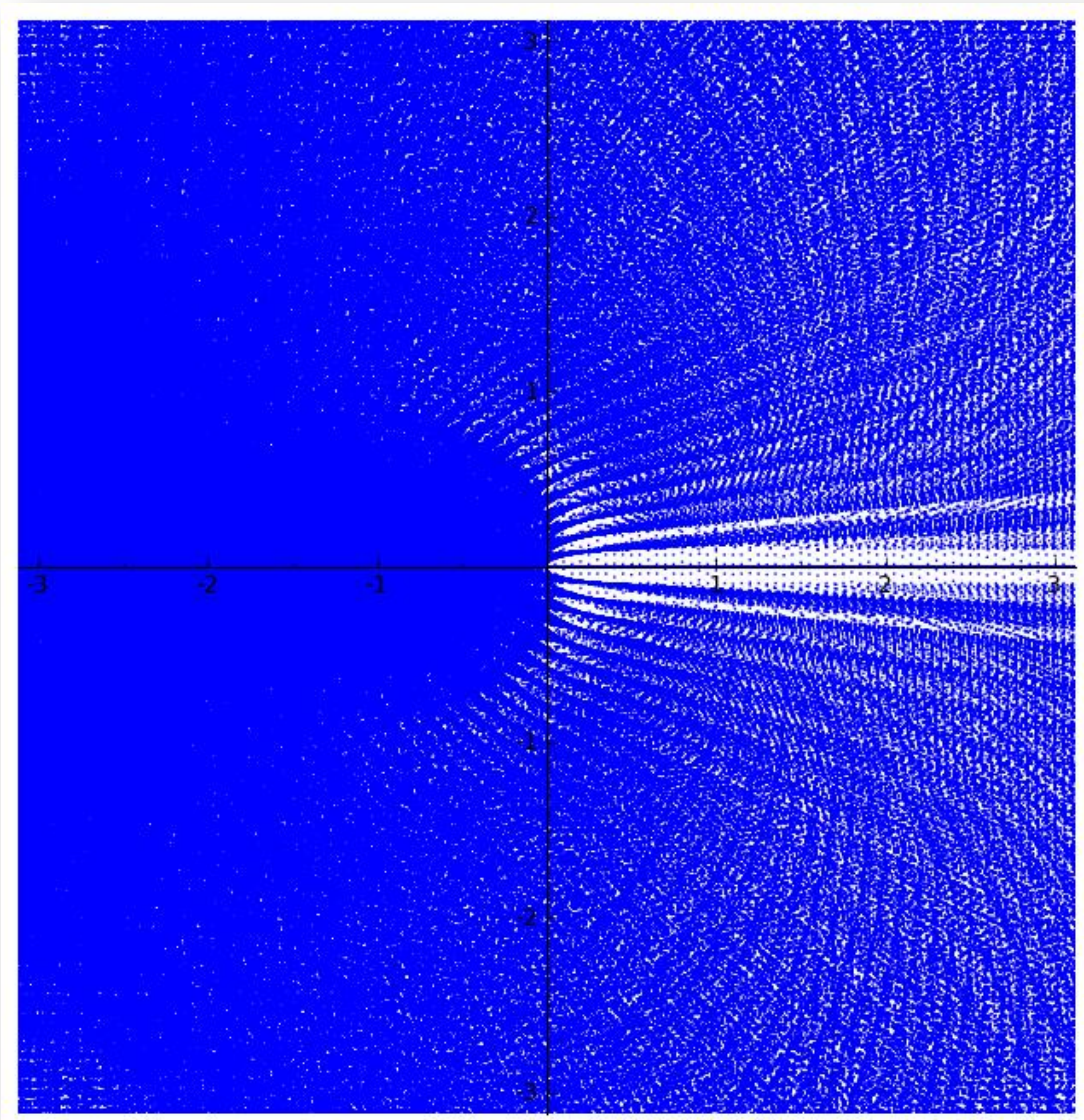


Zooming in about the origin:



Looking closer, we can see how non-uniformly positive numbers and negative numbers behave.

The same graph done with more points:

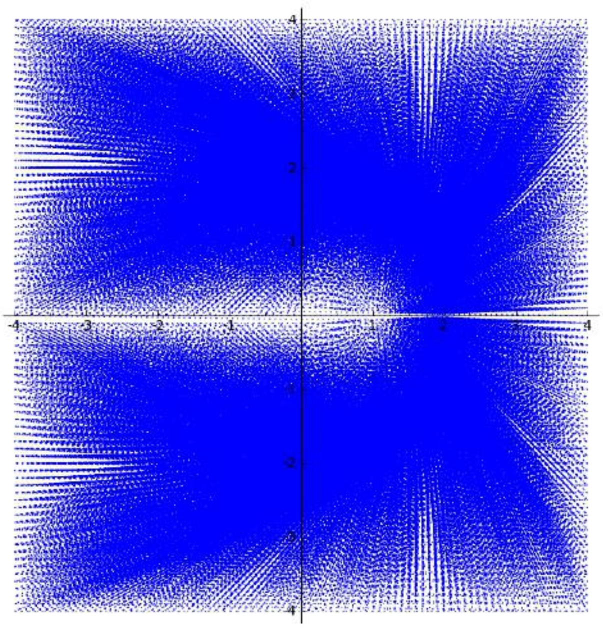
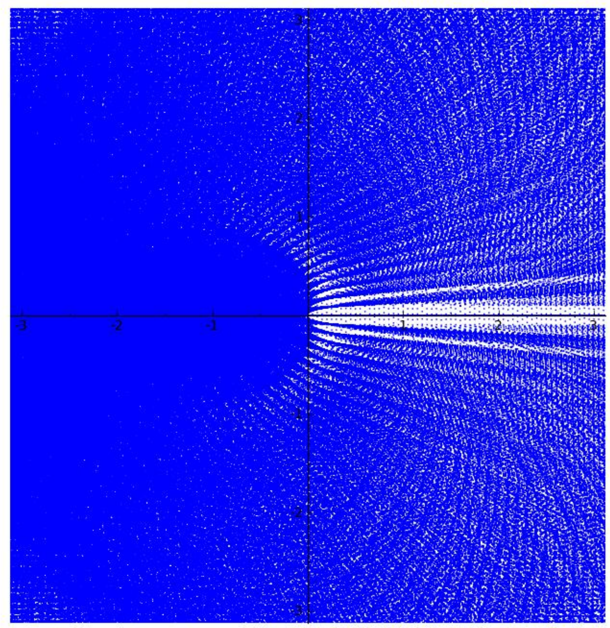


We are clearly "over-exposing" our graph at this level of detail, but we see how the real positive axis almost repels activity.

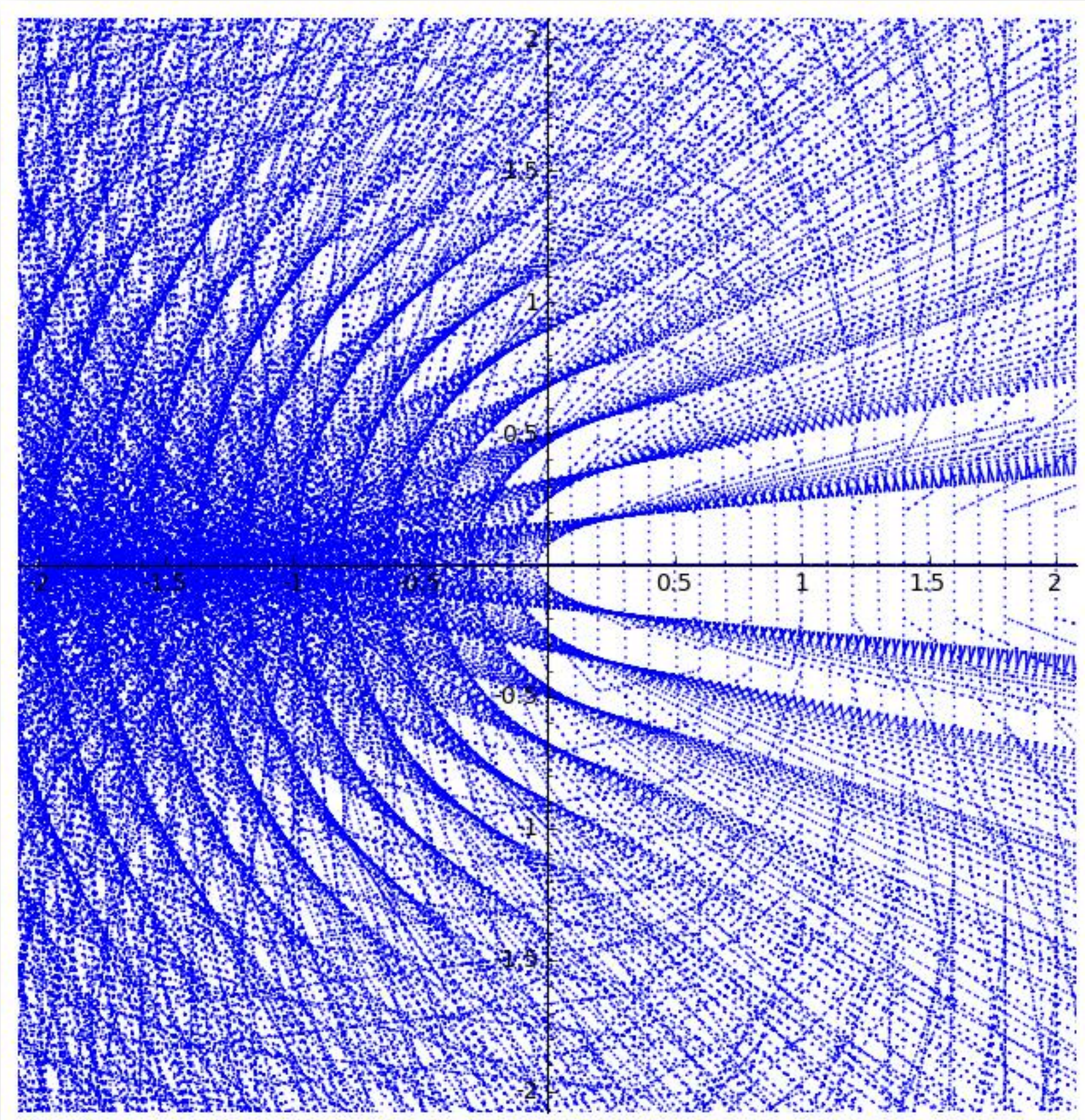
A Look Back:

Unlike square roots which seem to emanate from the negative real axis, squares seem to favor the positive real axis.

There is one other major difference. The graph of square roots in the complex plane (left graph) seems to repel the negative real number up to zero. Then, on the positive real axis, we still see it repel numbers from 0 to the number 1. This is asymmetric to the graph on the right where only the positive real axis repels values.

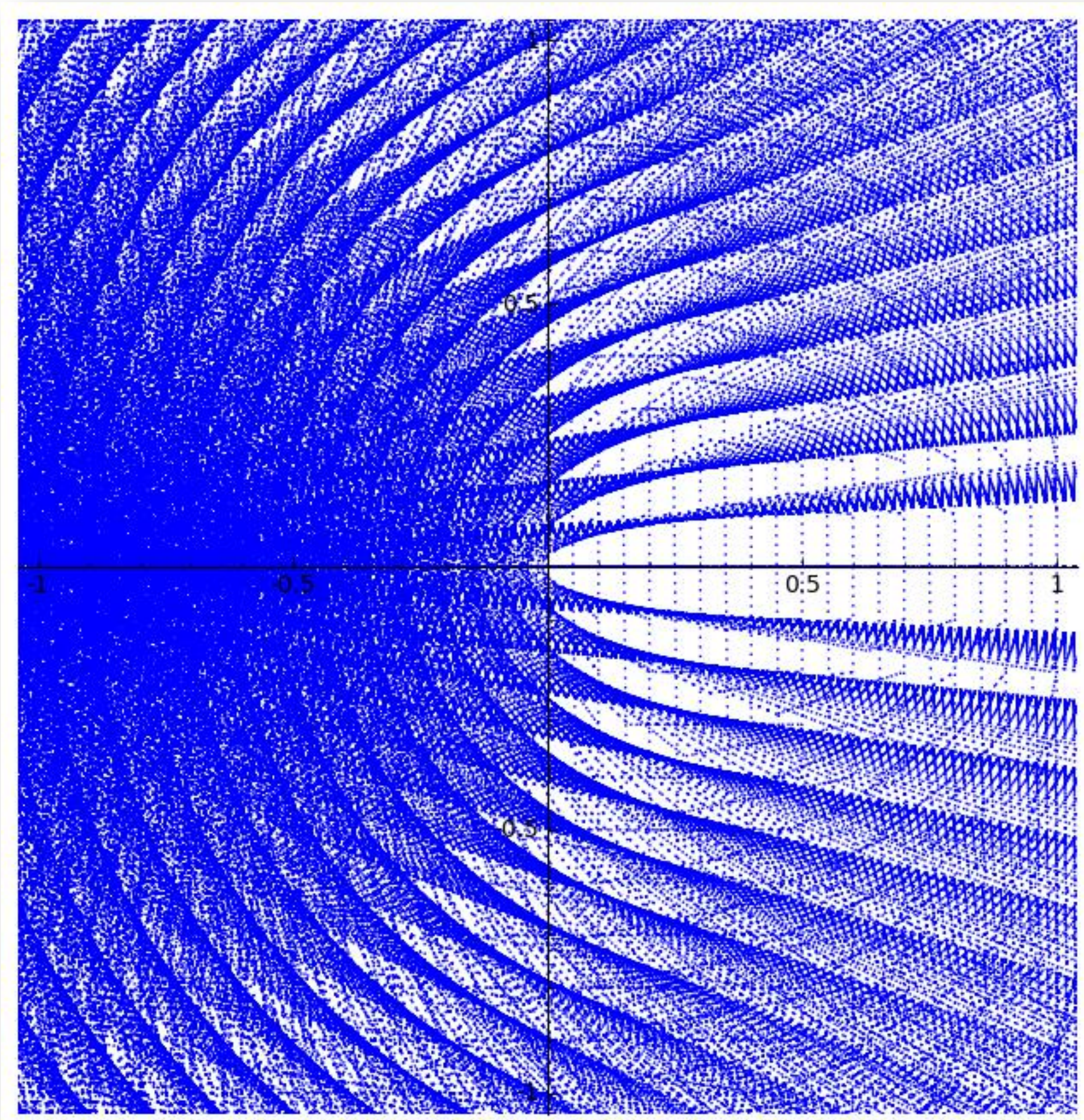
$X^{1/2}$	X^2
	

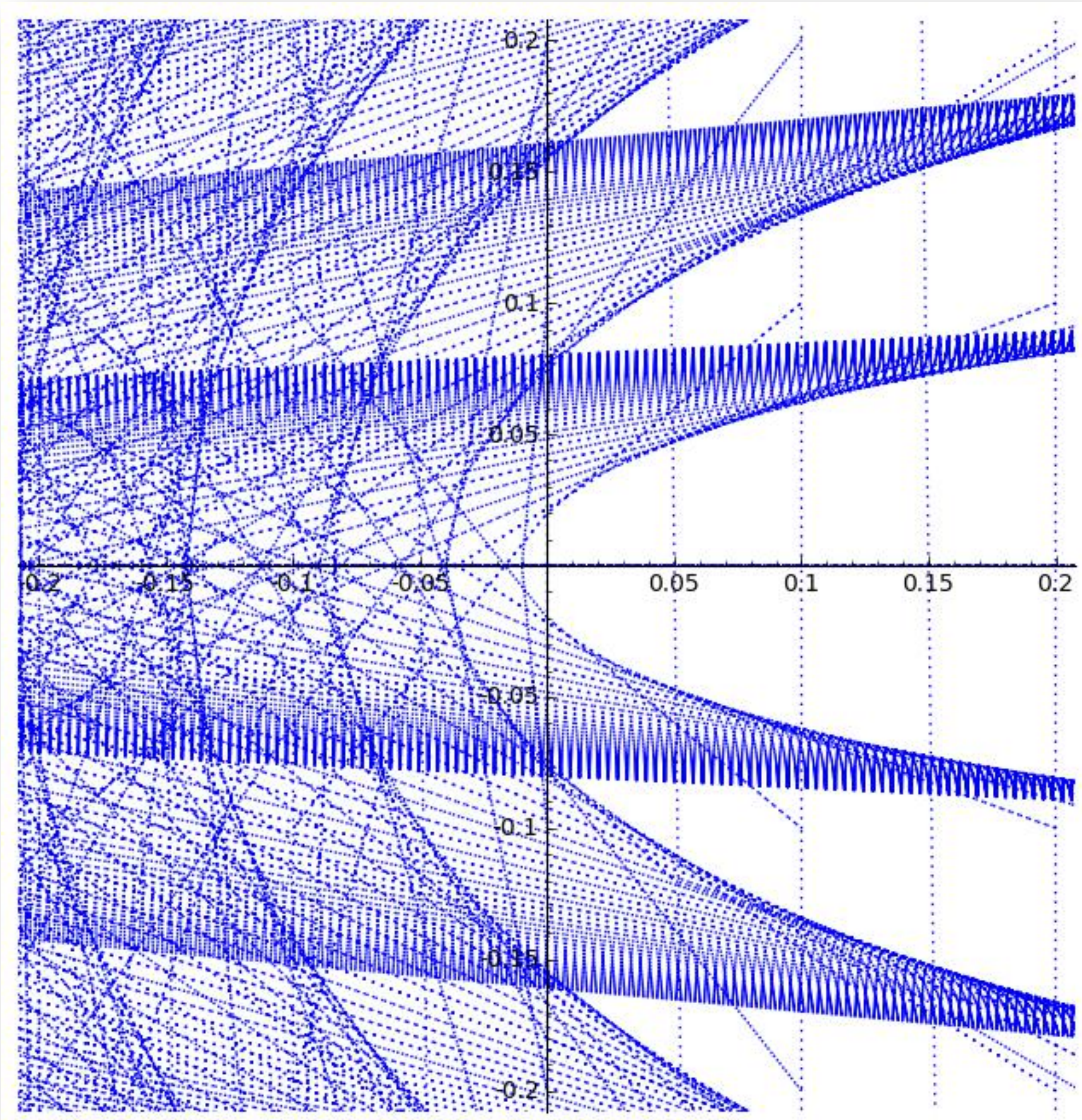
The following graphs show more detail regarding this field.



The positive numbers almost seem to “repel” anything from themselves.

X^2 (High Fidelity Image)



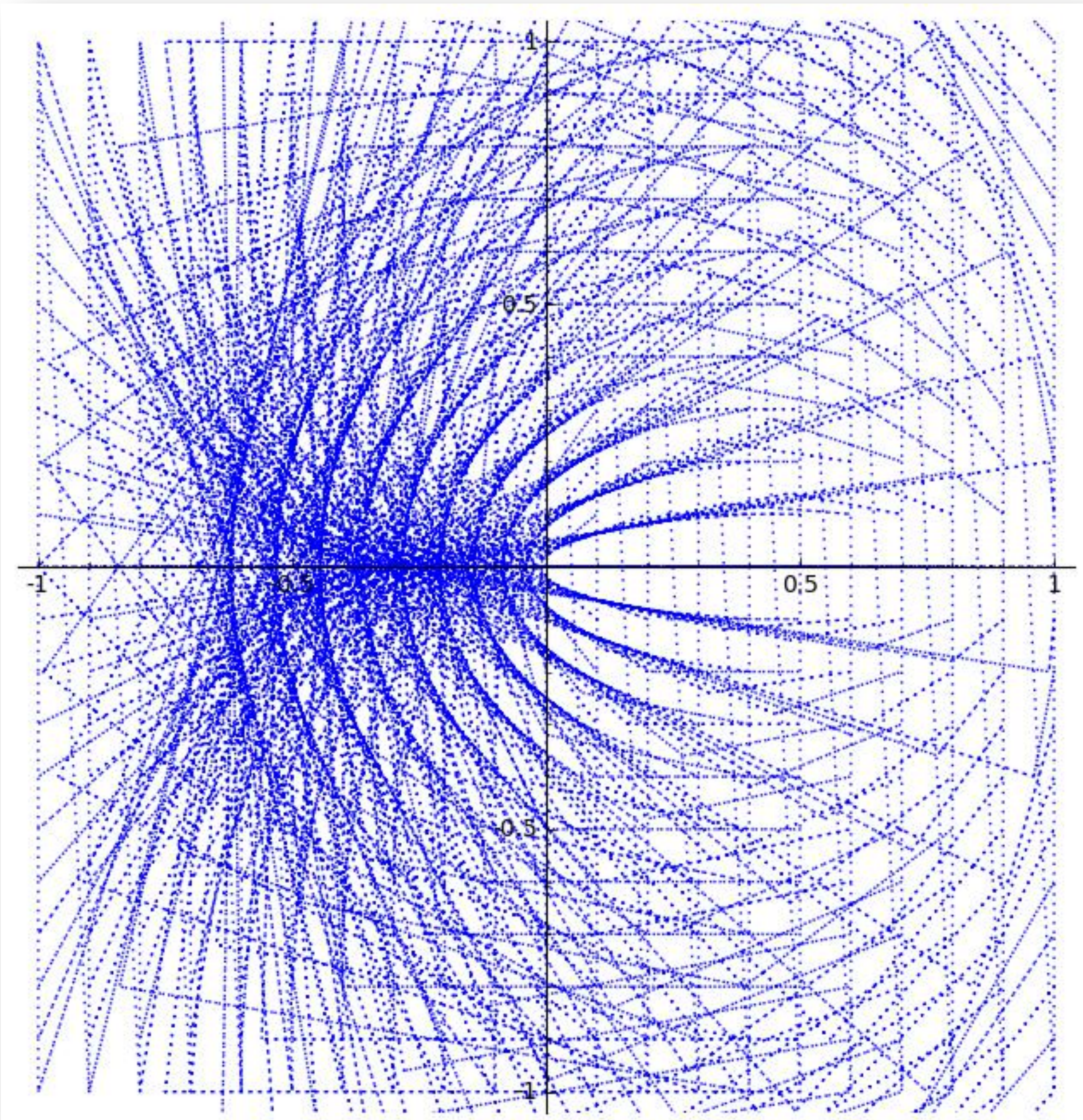


Even on the micro scale, we form these little rings on the negative axis.

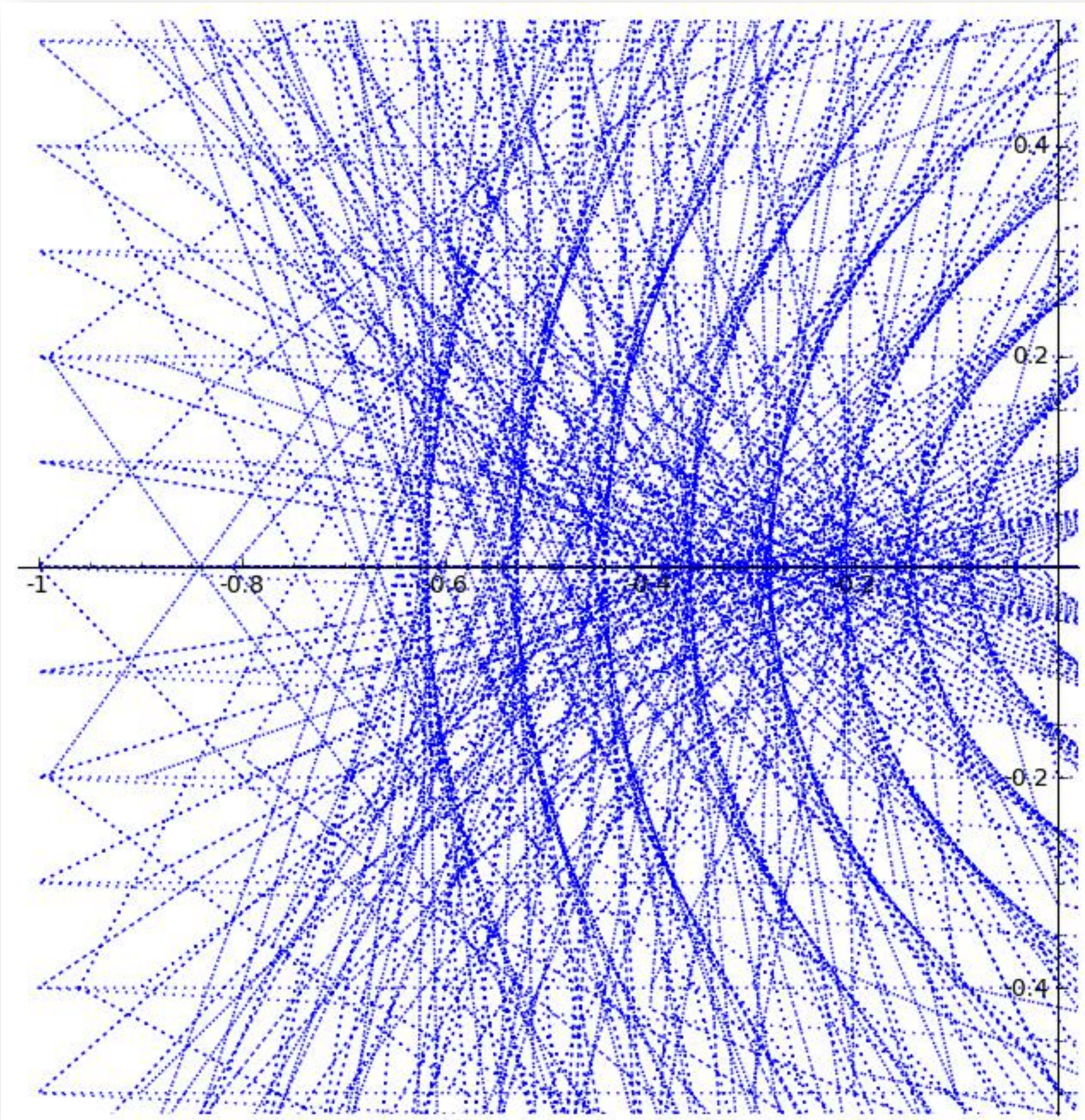
First question with infinite squares...

Is there any more we can learn from these small discs in the negative quadrants? Let's re-examine them by investigating the square function (i.e. X^2) using a 1X1 Grid.

X^2 (generated from a 1X1 Grid)



X^2 (1X1 Grid, 2X Magnification)



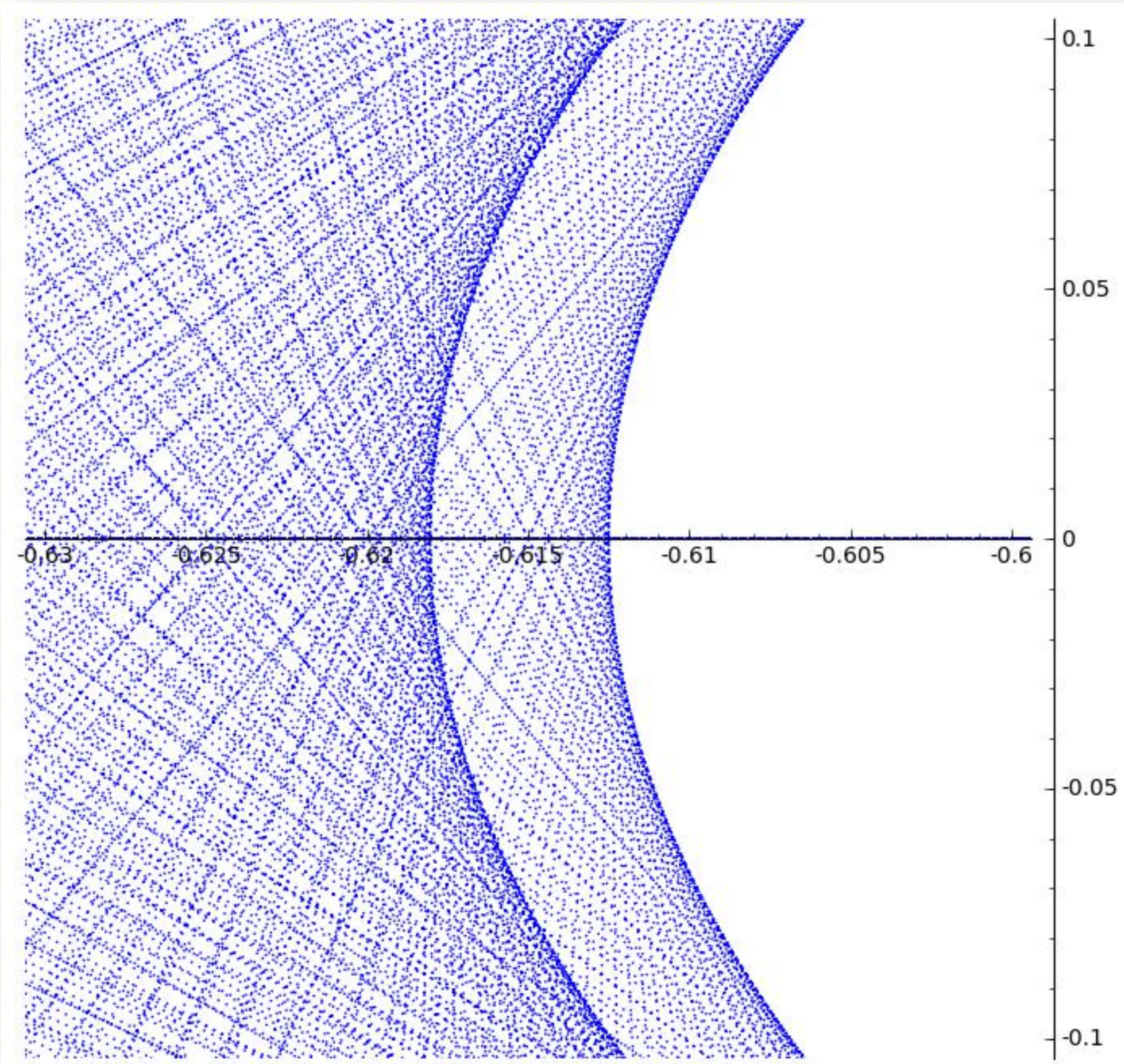
Question:

Why do the rings seem to end at a certain point?

What is that point?

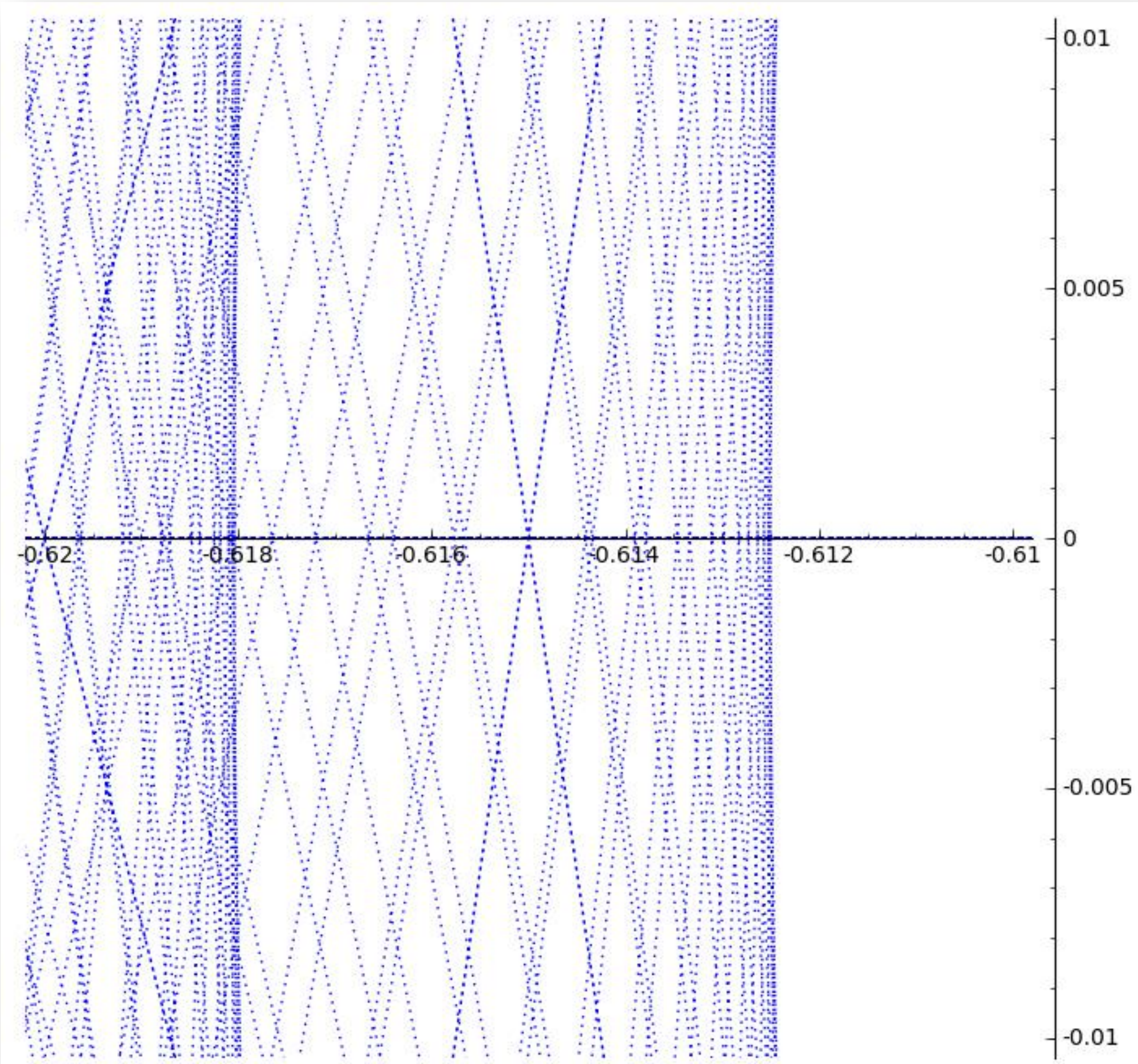
We see the familiar behavior with the two discs. Using extreme magnification we can "zero-in" on exactly where this disc exists. We will see that this outer value is actually the inverse of the **Golden Ratio!**

Zooming in on the X axis...



The following graph furthers our estimates by adding more precision.

Confirmation of 1/PHI



$$1/\phi = 0.618033989$$

The left bound is always 1/phi while the right side keeps shrinking and getting closer to the left. I have verified this to 8 decimal places just by zooming in graphically.

Adding to the List…

We can see how most of our mathematical constants are in a way children of infinity.

It's time to update our list of mathematical constants we've come across. We can now add the Golden Ratio to the list.

List of related concepts discussed so far:

- ✓ Square Roots
- ✓ Imaginary Numbers
- ✓ Pi, and therefore e
- ✓ The concept of Zero and Infinity
- ✓ Randomness
- ✓ The Feigenbaum Constants
- ✓ Light / Optics (in a small way)
- ✓ **Phi, the Golden Ratio**

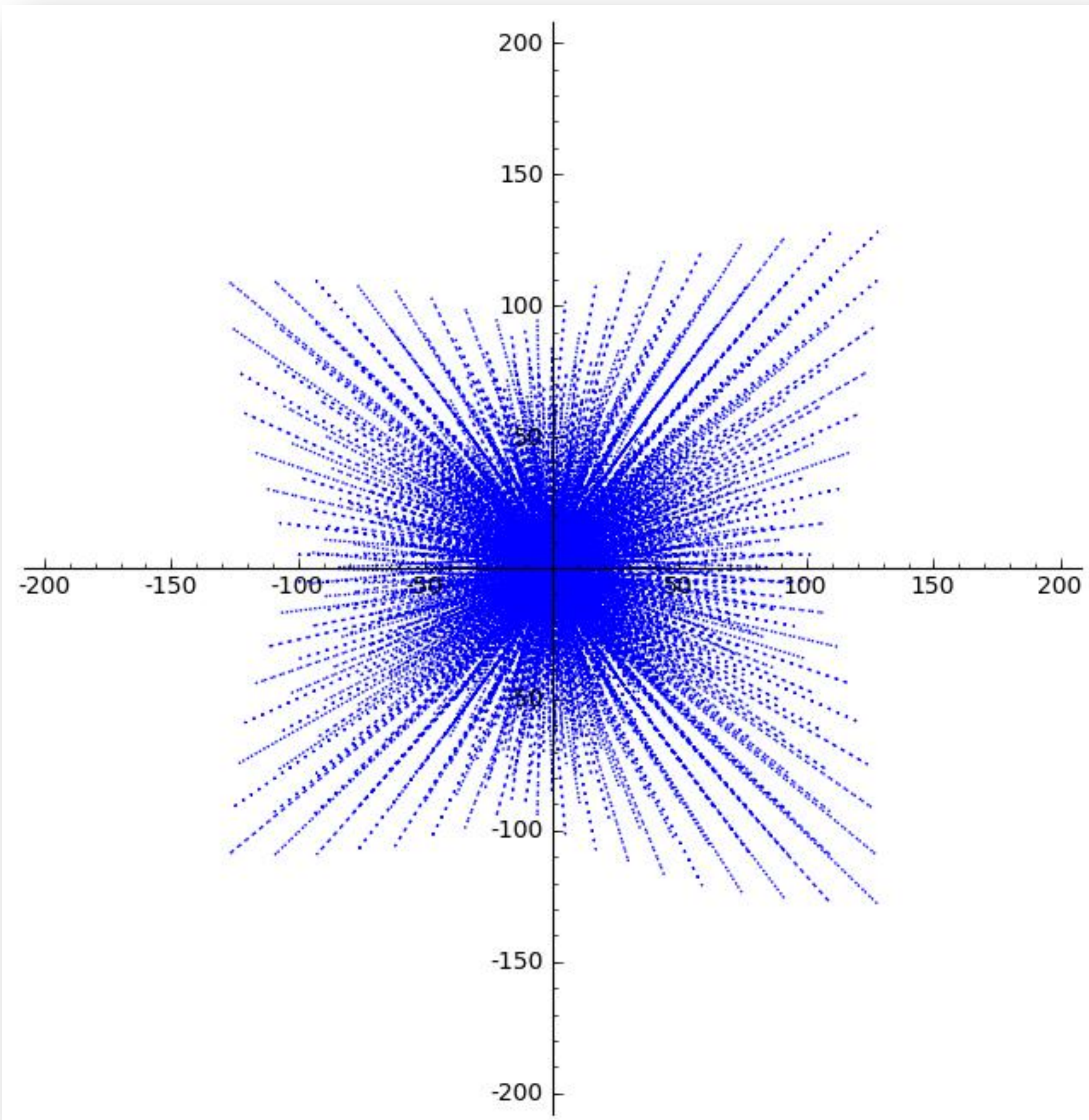
There still many more patterns to uncover.

Higher Order Powers: Beyond Pretty Pictures

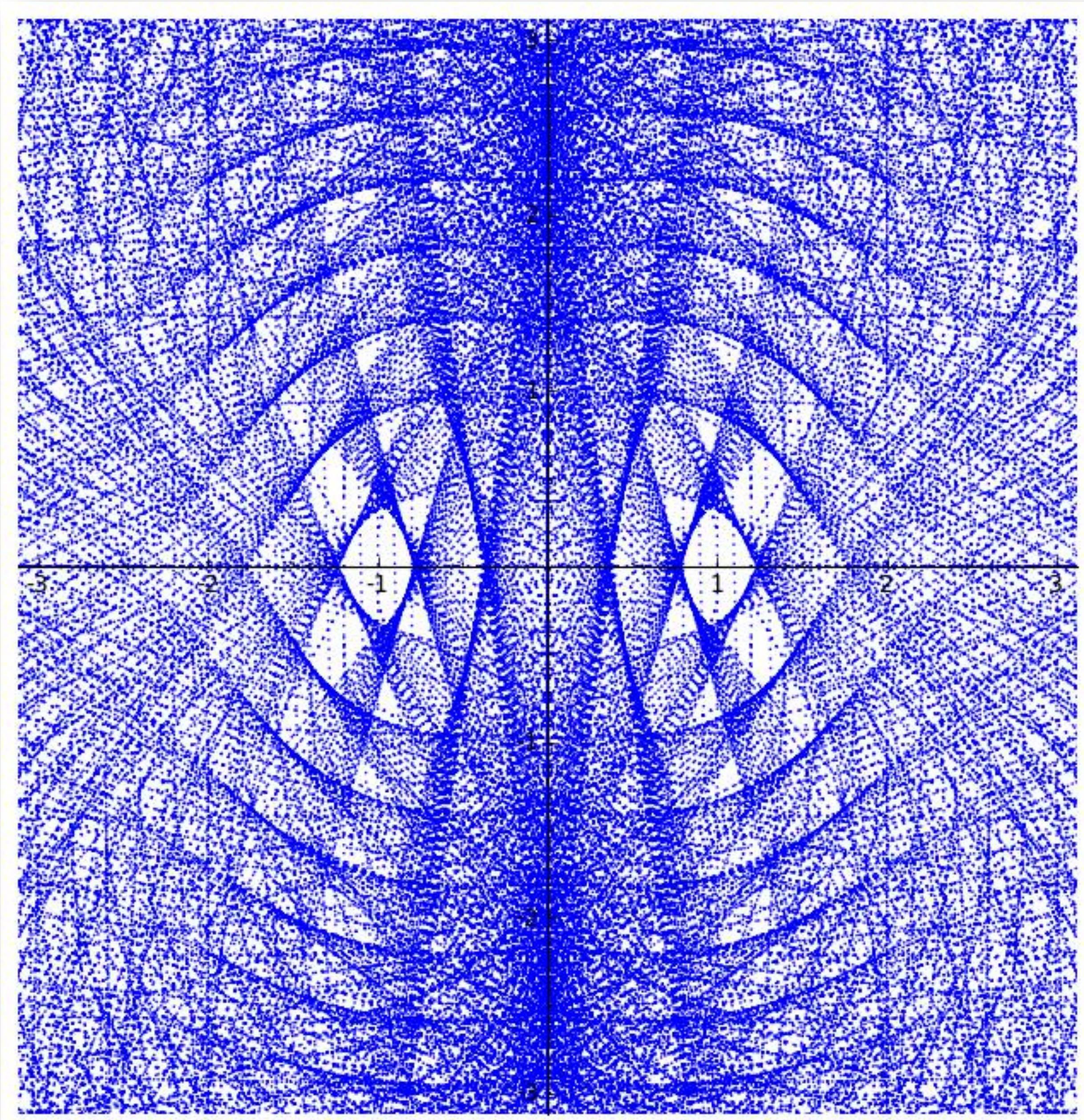
In this next section, I will ask you a simple question:

Do you notice any patterns?

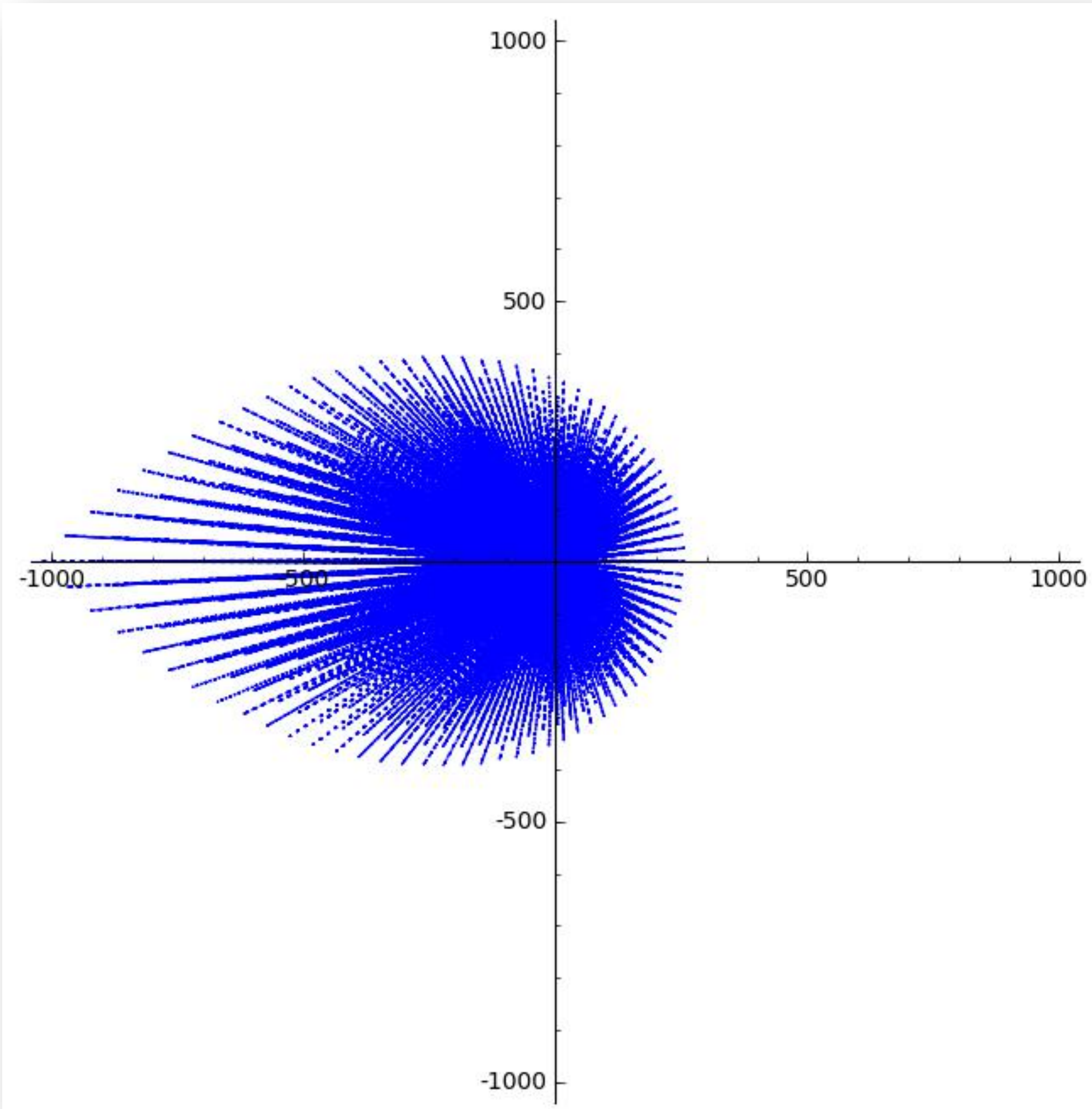
x^3



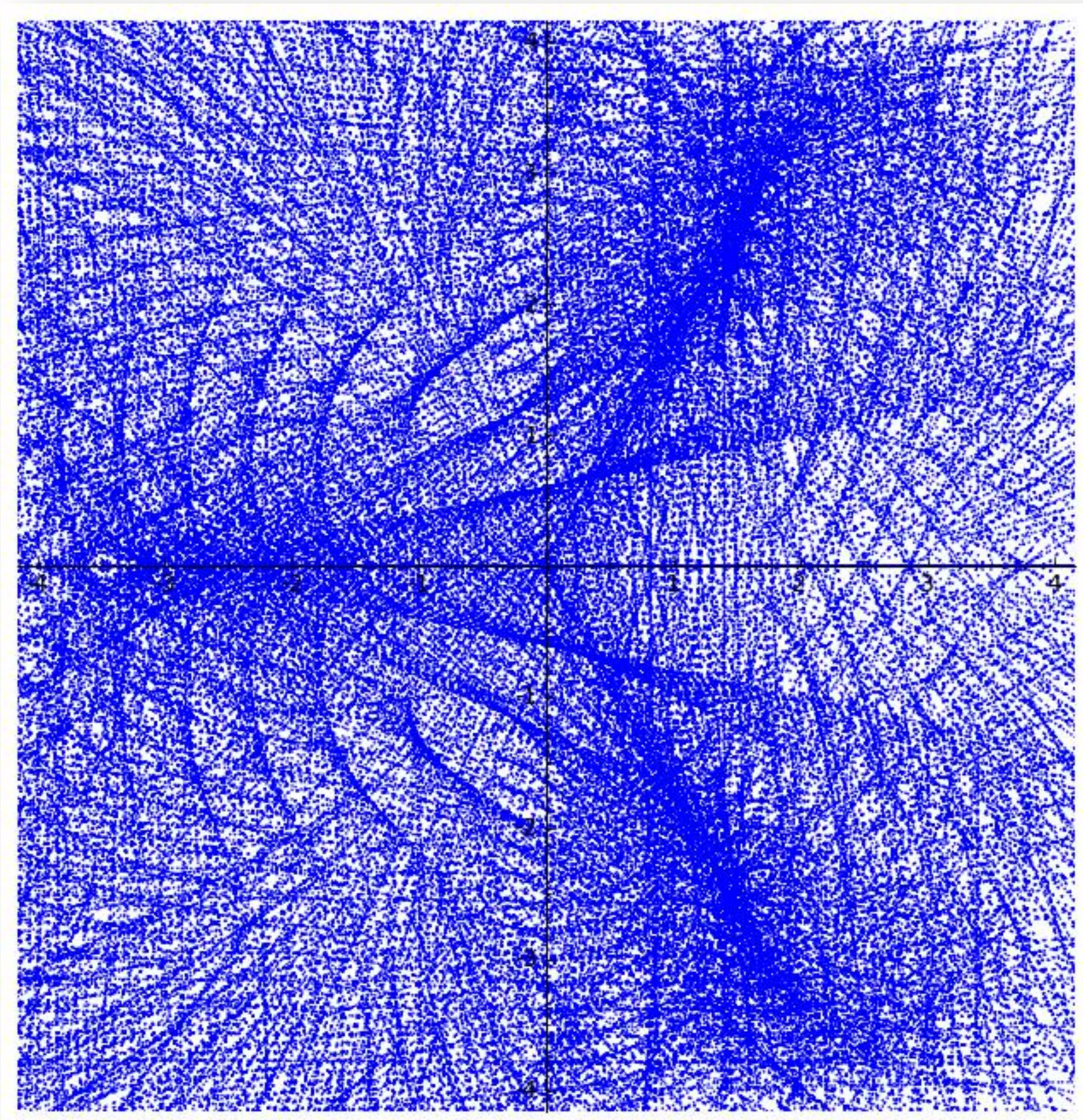
x^3



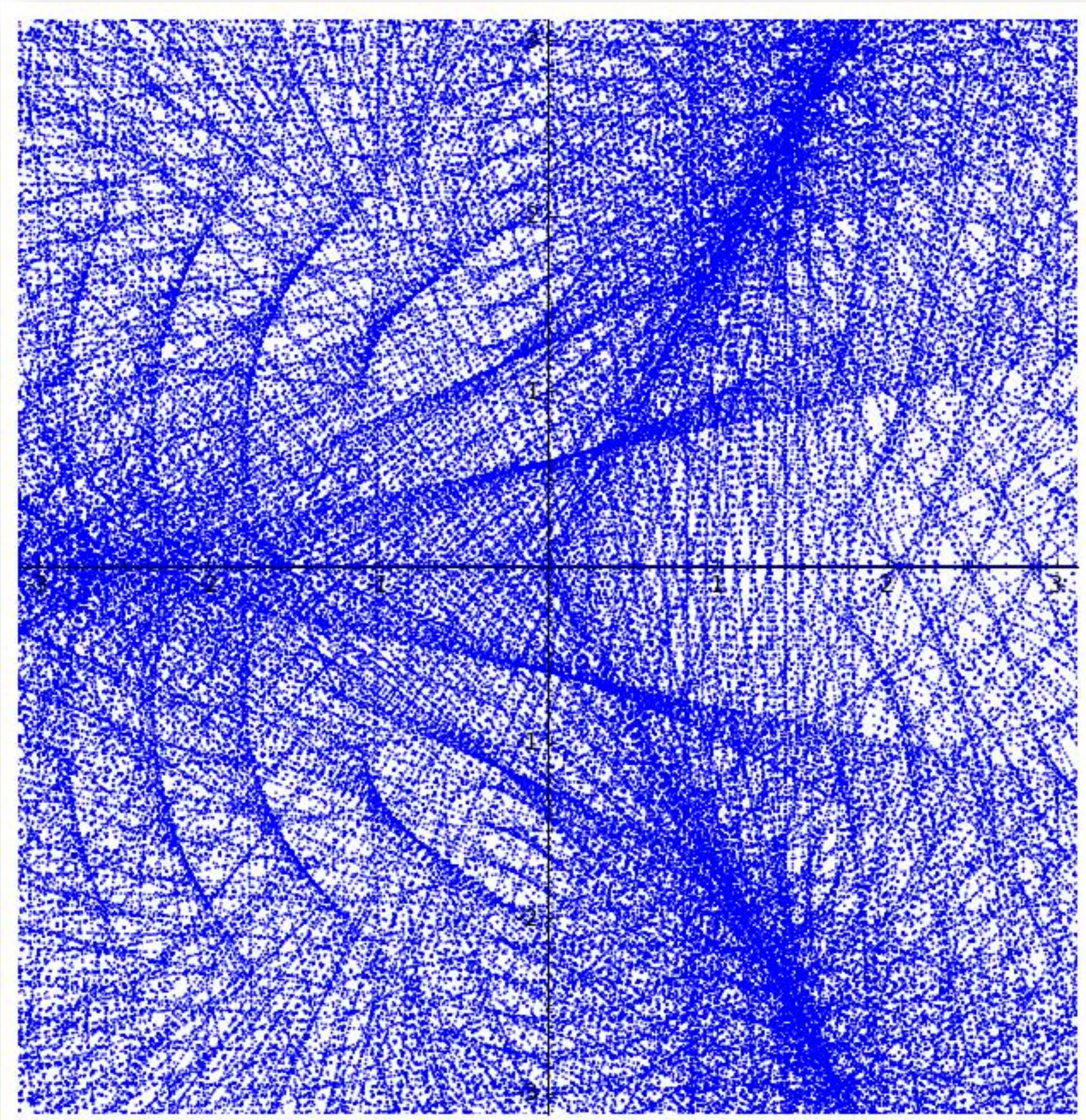
x^4



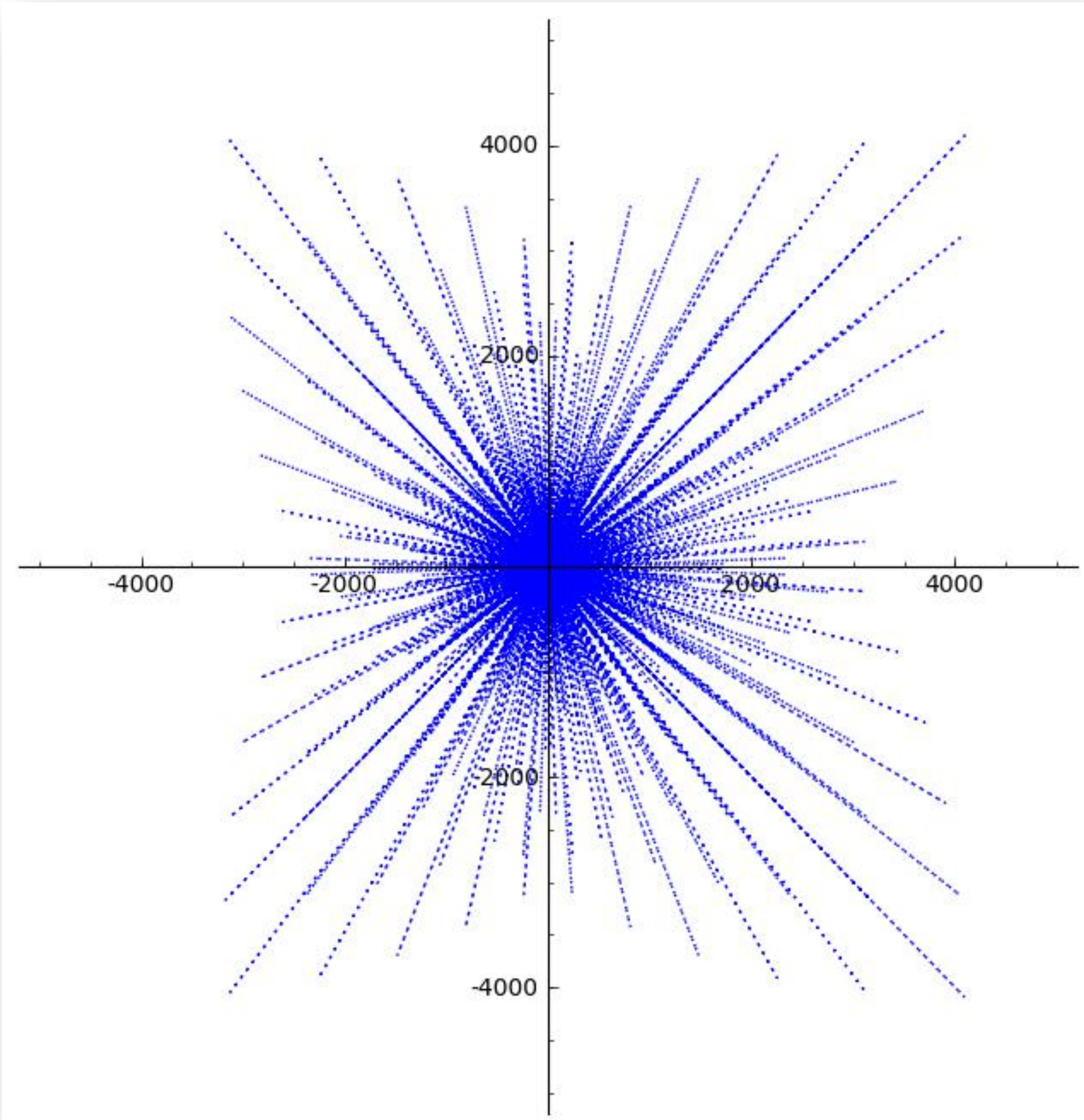
x^4



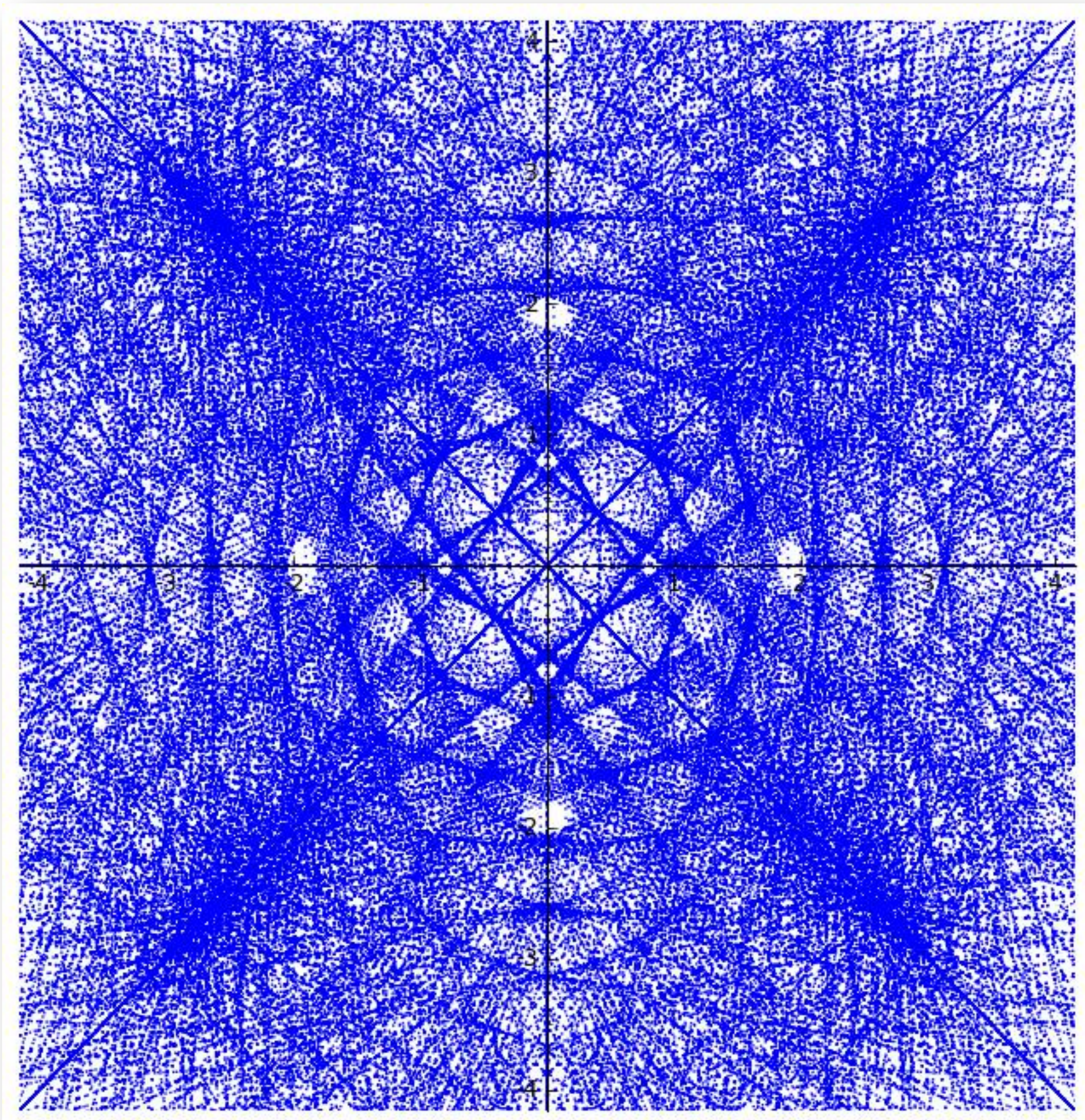
x^4



x^5

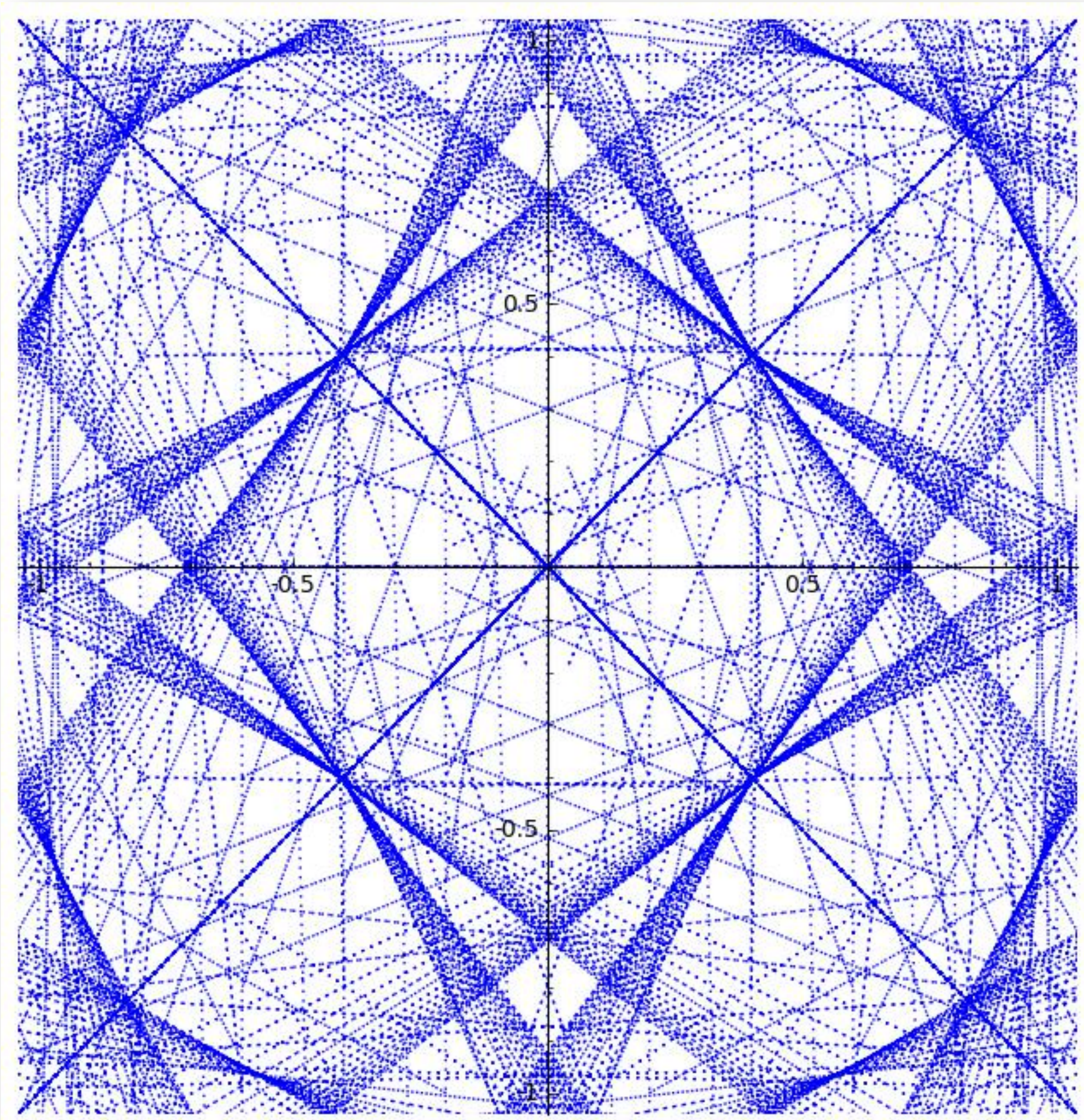


x^5

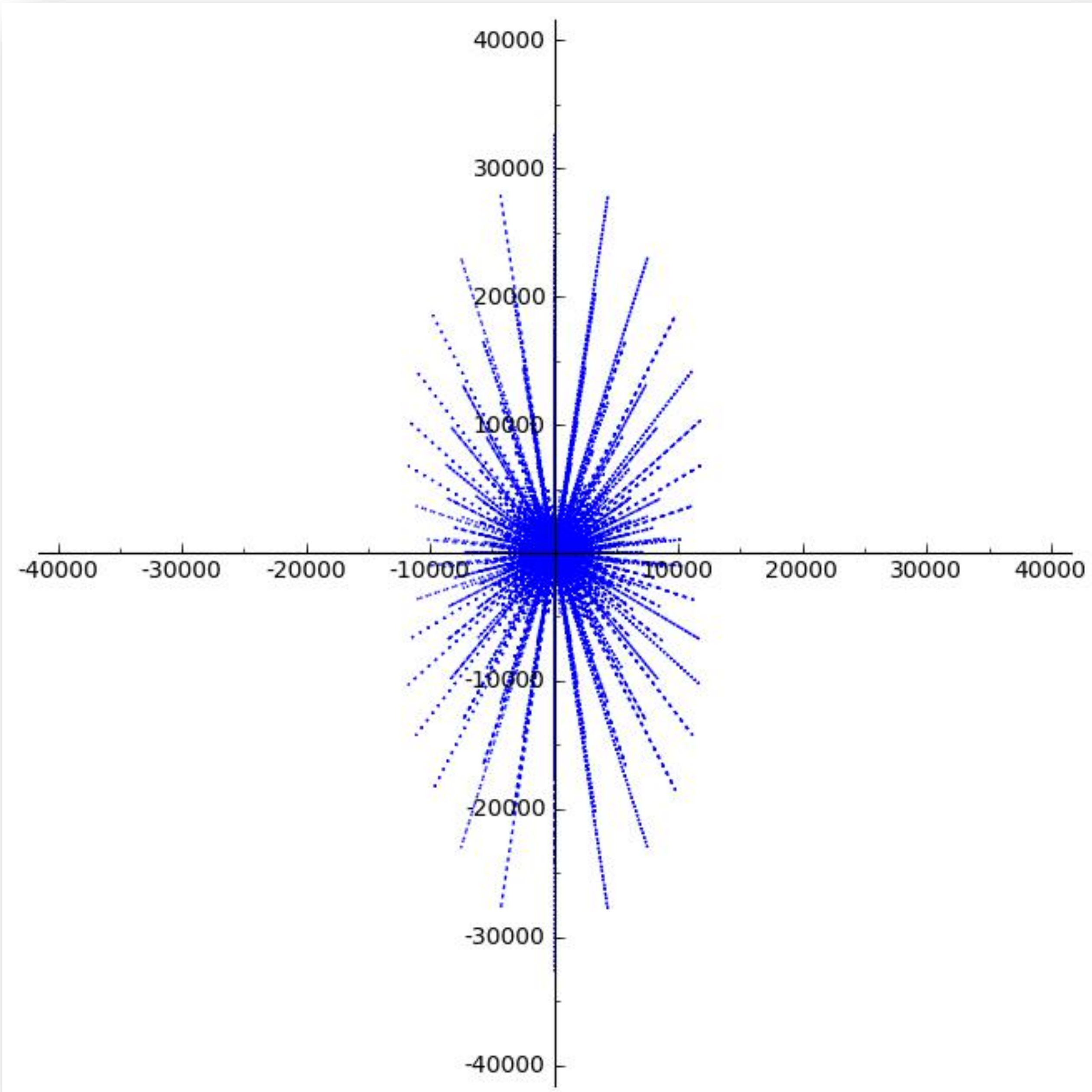


It's amazing to marvel at the increasing complexity of the inner shapes.

X^5 (Zoomed In)



x^6


40000
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20000
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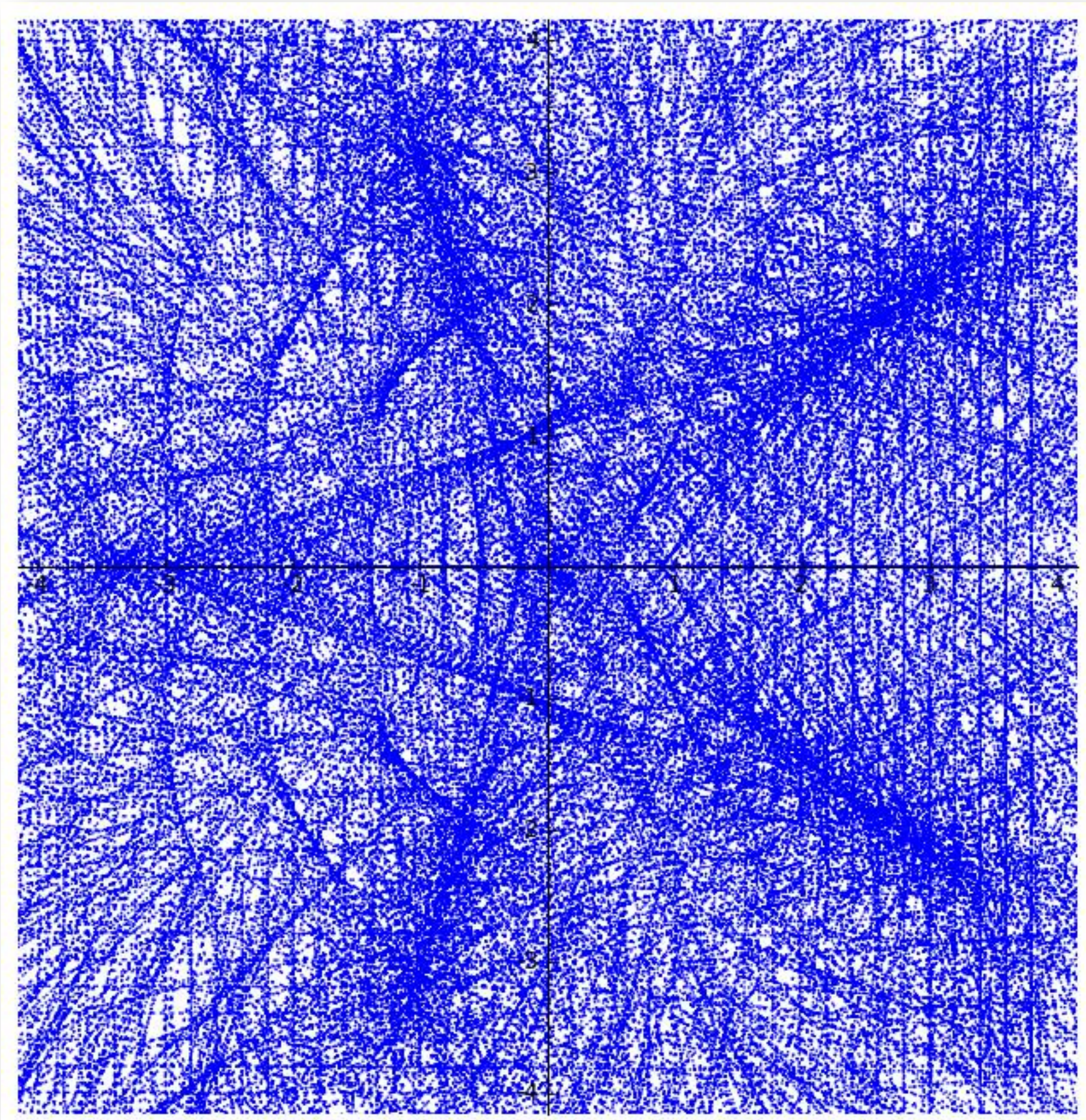
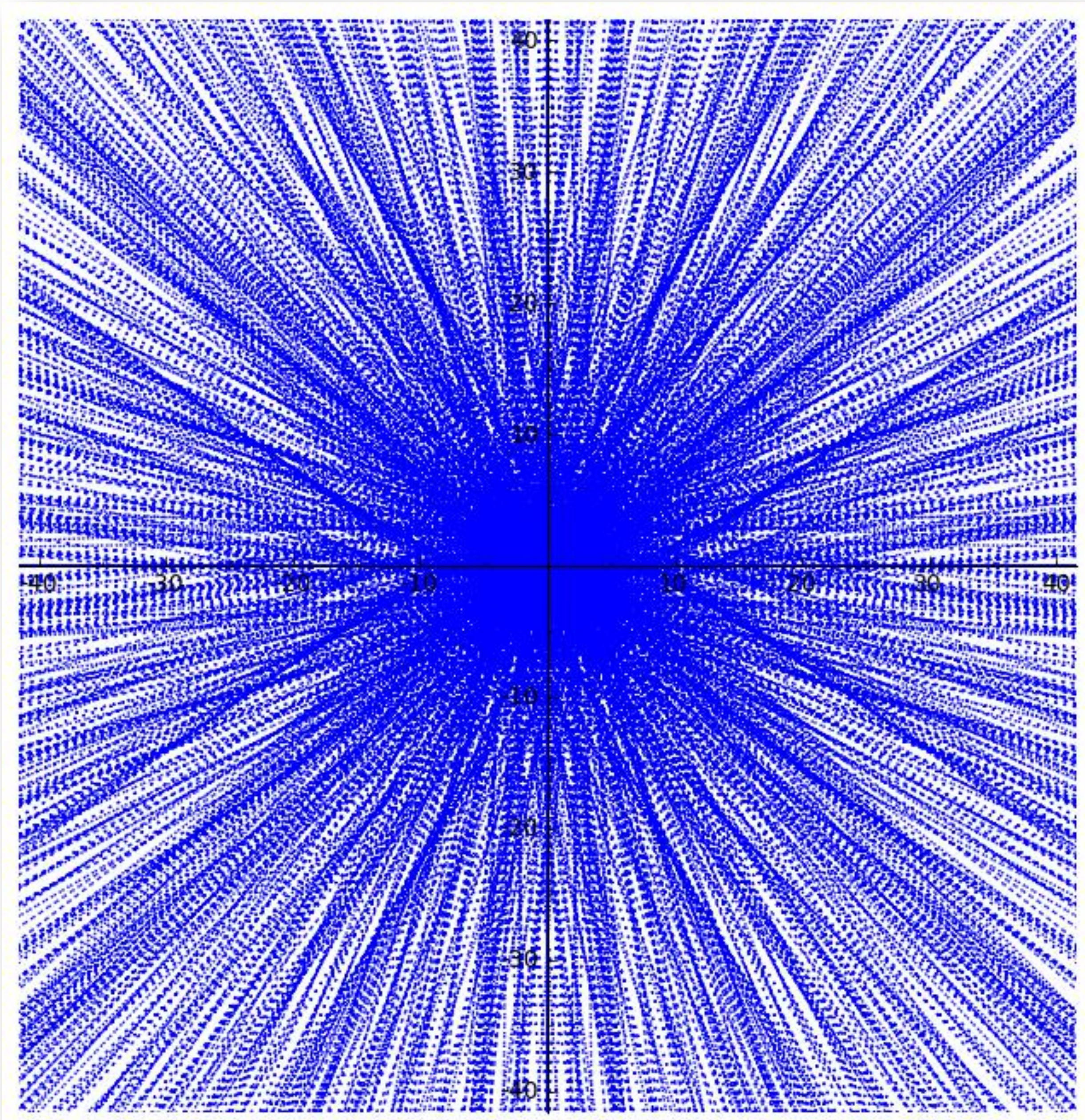
-20000
-10000
10000
20000
30000
40000

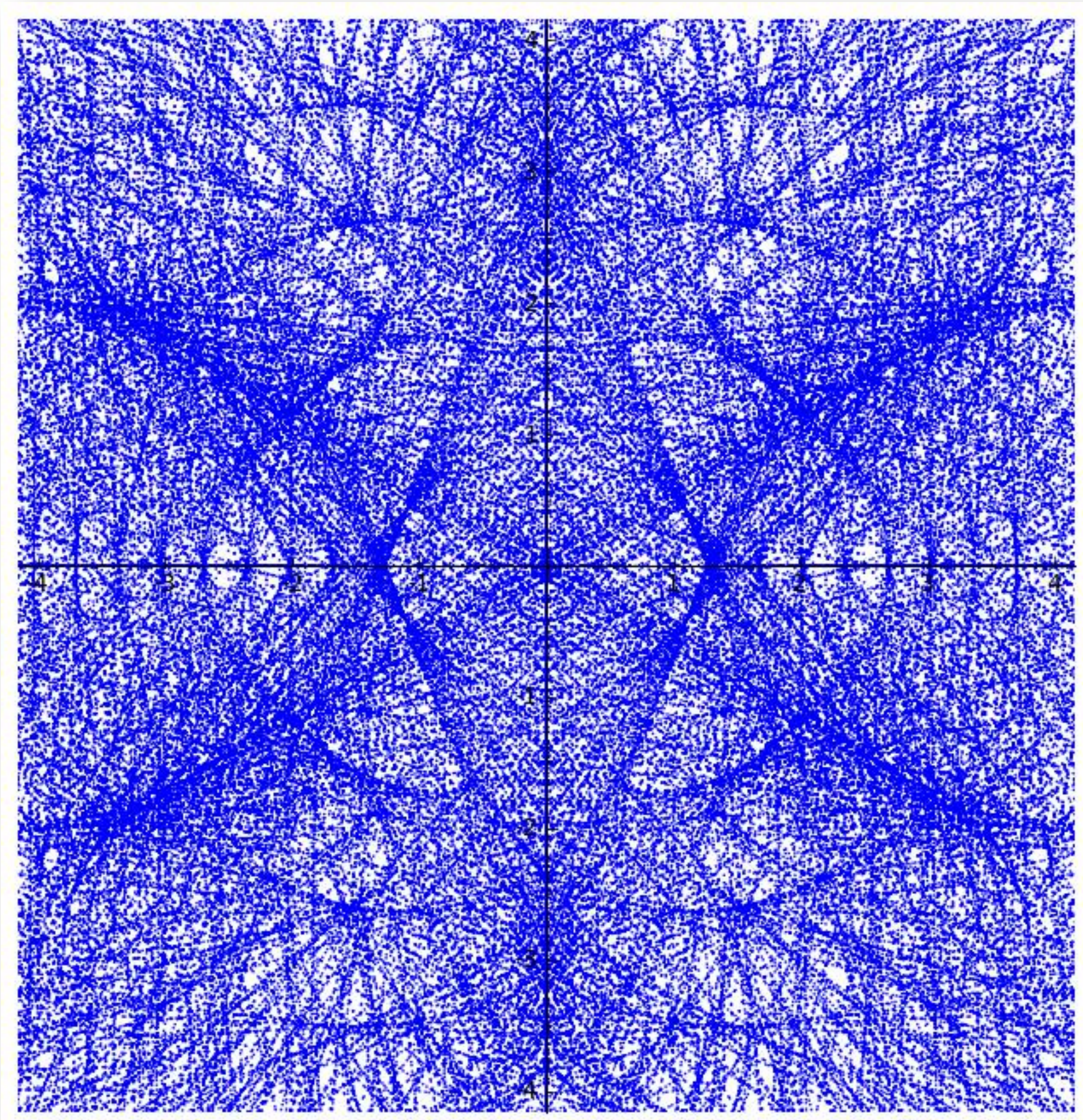
x^6



x^7

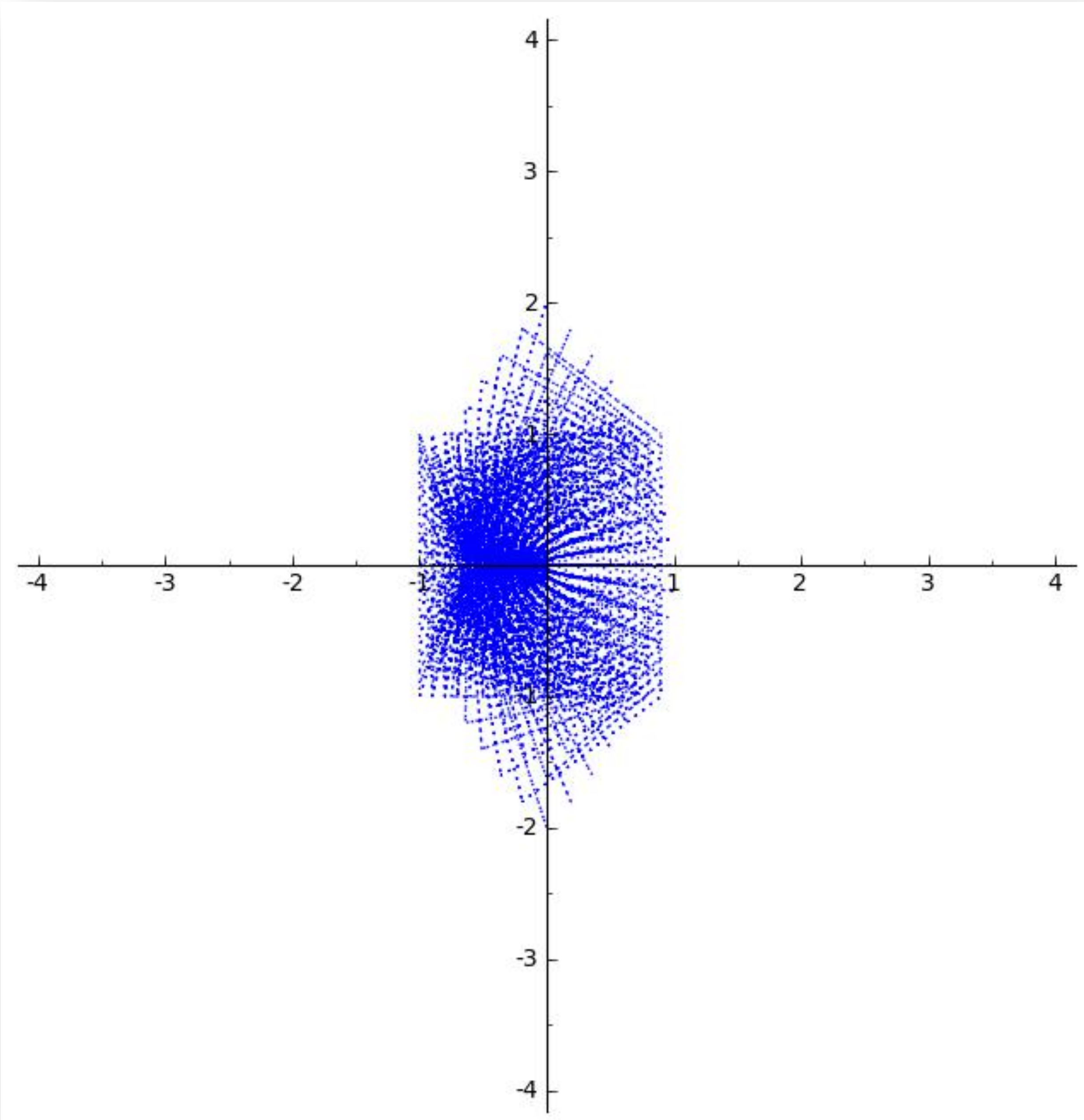


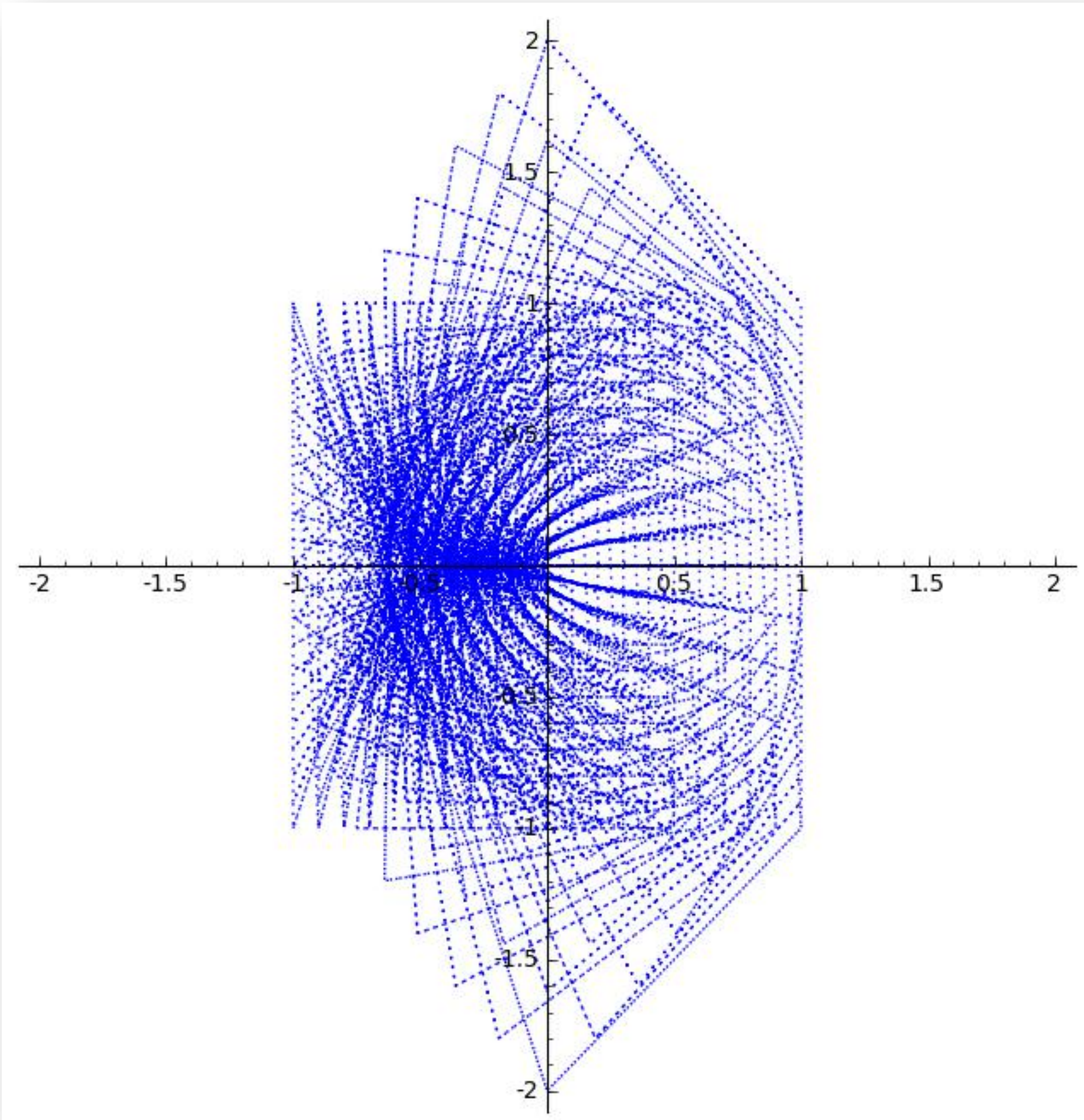
x^7



1X1 Boxes

Using a box of 1X1 for X^2




2
1.5
1
0.5
-0.5
-1
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-2
-2
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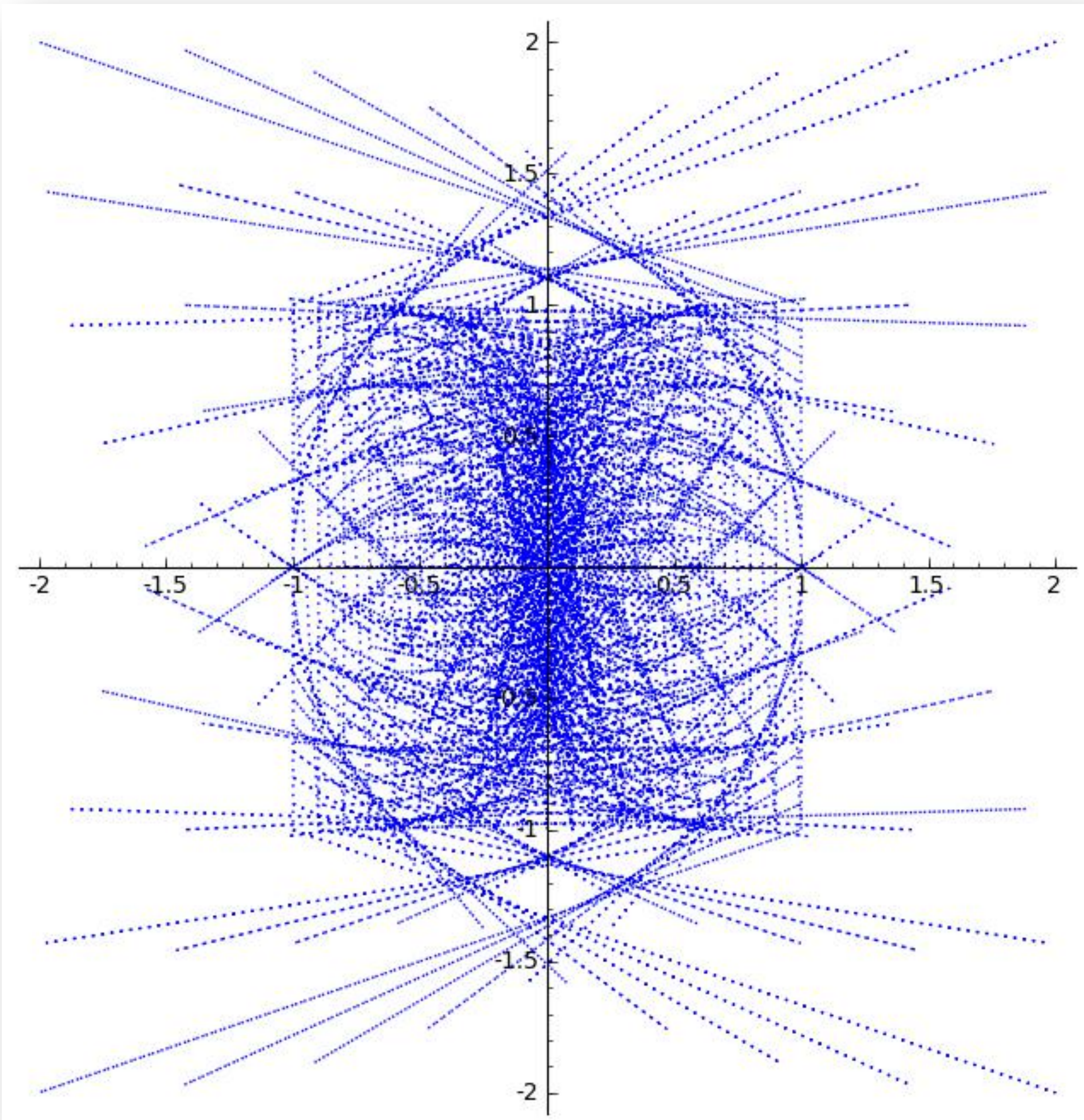
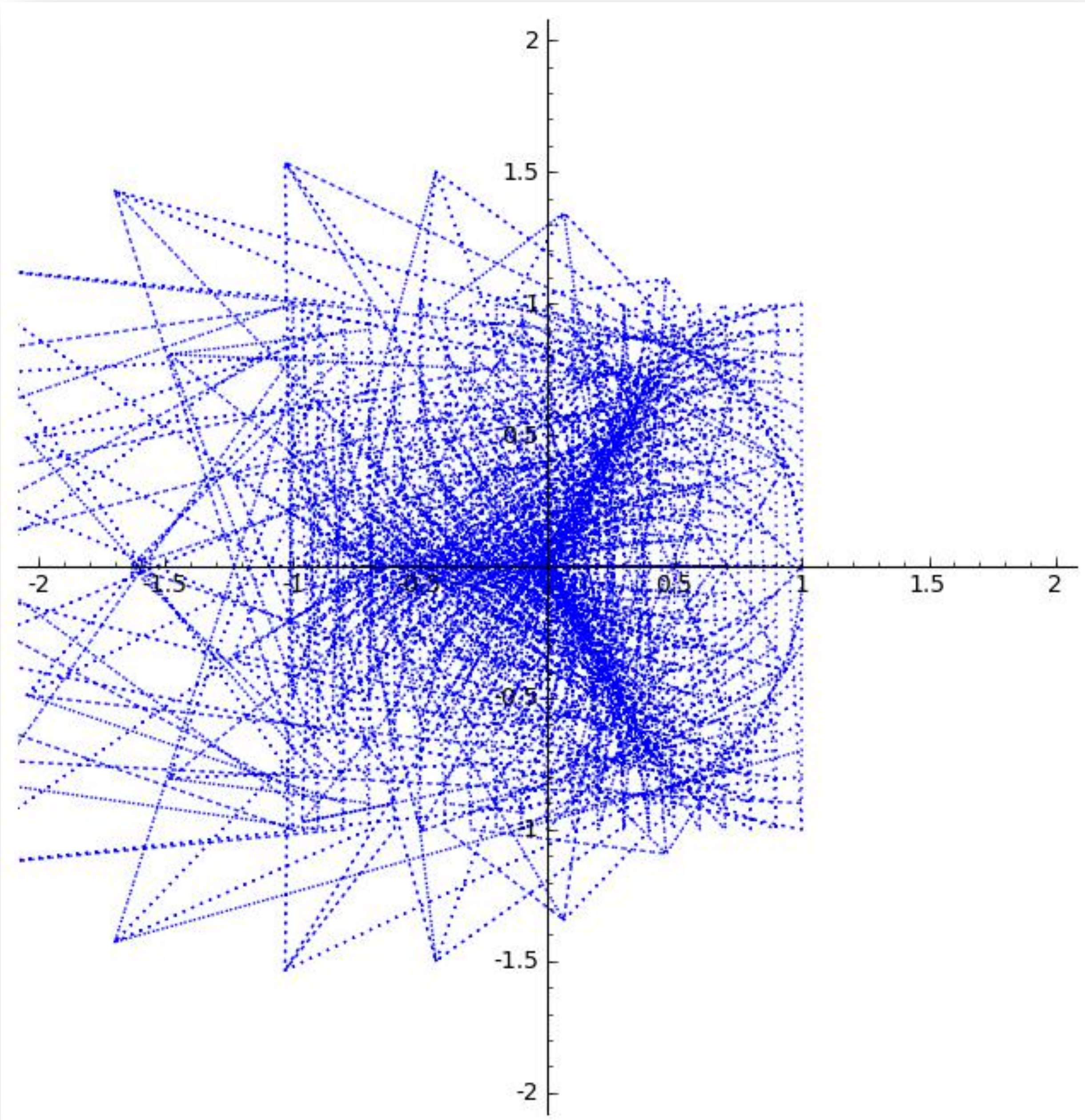
-1
-0.5
0.5
1
1.5
2

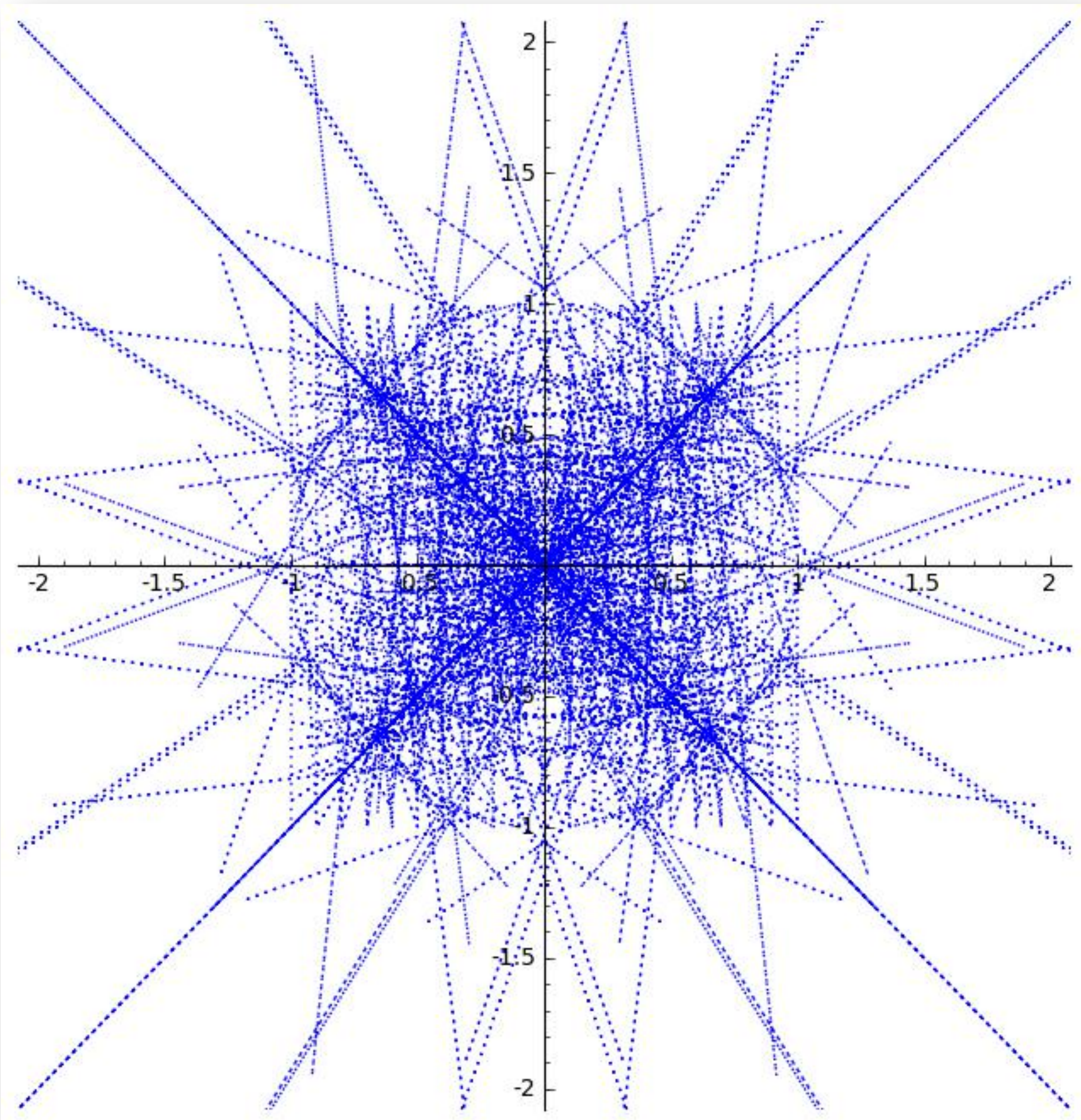
Using a box of 1X1 for X^3



Using a box of 1X1 for X^4



Using a box of 1X1 for X^5



Aside:
After 5 iterations you need to expand your inner box to properly visualize the inner figure.

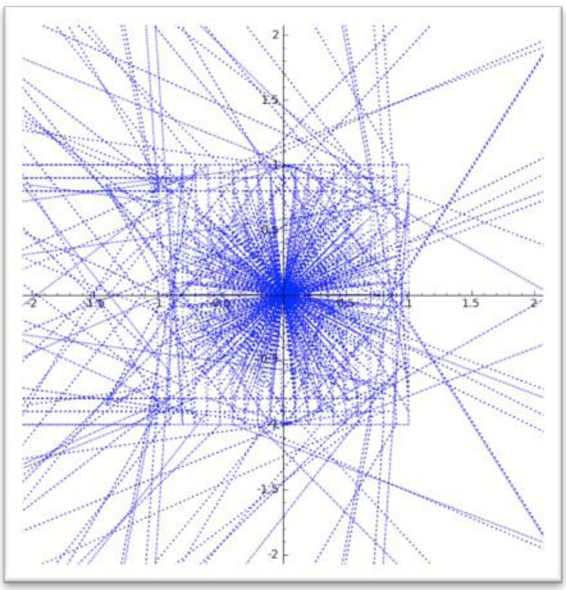
Did you notice any patterns?

The table below summarizes the important patterns that have emerged. The exponent operation seems to always create an object with one less *spoke* than the exponent taken. For example, the X to the 3rd operation will create a shape with 2 spokes, whereas X to the 4th, will create a shape with 3 spokes.

Function	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^n
Number of "Spokes"	1	2	3	4	5	6	7	n-1

You should immediately notice the parallel to the infinite root exercise. Infinite roots form a circle pattern that has n-1 petals while taking the infinite power creates a shape with n-1 spokes.

I'm sure if we had the patience to count, we would count 99 spokes in this graph where the function is X^{100} :



It's time to revisit our previous table regarding the infinite root and the circle within leaves. The table is provided again:

Root	2	3	4	5	6	7	8	9	∞
# of Leaves	1	2	3	4	5	6	7	8	∞
Shape Area	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{4\pi}{5}$	$\frac{5\pi}{6}$	$\frac{6\pi}{7}$	$\frac{7\pi}{8}$	$\frac{8\pi}{9}$	$\frac{9\pi}{10}$	1

An attempt to combine tables…

Function	X^0	$X^{1/5}$	$X^{1/4}$	$X^{1/3}$	$X^{1/2}$	X^1	X^2	X^3	X^4
# Patterns	∞	4	3	2	1	0	1	2	3

Chapter 4: Infinity

Introduction

So far, we talked a lot about the concept of Zero, or the *origin,* as mathematicians like to refer to it. We have been looking very closely at the *origin* and doing lots of fancy operations near the *origin* such as taking square roots over and over, performing division and using exponents.

But, we have yet to examine the other end of the spectrum: when the solution becomes continually larger and larger. Mathematicians have called this concept, *infinity*. But what is *infinity*?

The goal of this chapter is to show that it is possible to "define" infinity. We will attempt to define different "classes" of infinity and show how they can be related to each other. In next chapter, we will be selecting one form of *infinity* to be the inverse of *zero.* Mathematicians have yet to define a specific form of infinity that represents the inverse of zero.

Flavors of Infinity

Are all concepts of infinity the same? No, there are many different ideas of infinity. Can we separate infinity into different classes and distinguish them from one another? For ease of reference, I have created some fairly simple names. Our goal will be to find some sort of relationship between all of the classes.

I have ordered (and named) some of the most commonly referenced infinite series'.

Forms of Infinity (ordered by growth rate)

$$\sum_1^{\infty}\frac{1}{x}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=I^{Weak}$$

This is also the Rieman Zeta Function using 1.

$$\sum_1^{\infty}1=1+1+1+1+\ldots=I$$

$$\sum_1^{\infty}x=1+2+3+4+\ldots=I^{Strong}$$

$$\prod_1^{\infty}2=2*2*2*2*2*\ldots=I^{Mult.}$$

$$\prod_1^{\infty}x=1*2*3*4*5*\ldots=I^{Factorial}$$

$2^{2^{2^{\cdots}}}$ Evaluated from bottom to top

$2^{2^{2^{\cdots}}}$ Evaluated top to bottom (called a ***tetration***)

What do we do next? How can they all equal each other? Additionally, **which one is the inverse of *zero***?

Mathematicians don't define which, if any, of these runaway series is the inverse of zero. We state that zero has no inverse. If that is the case, then zero is infinitely "powerful". It's counterpart, *infinity* would also be *infinitely* powerful and instantaneously reach some maximum limit. This model implies that there is a largest number in the set of real numbers. This is a contradiction to our definition of real numbers.

This alone should force us to decide on some inverse of zero. Perhaps we need to define various flavors of zero. If that is true, how many flavors of zero are there? Are there an infinite number of both *infinity* and *zero*? At this point, we've created a lot more questions; let's see if we can't start digging for some answers.

Methods to Unify the Growth Rates of Infinity

Using modern computing, we can perform iterations and see how these different forms of infinity compare to each other. Our goal will be to relate all of them together.

We will put each form of infinity in the form of the slowest growing form, I^{Weak}. Just doing this simple exercise uncovers a few more fundamental mathematical constants. We will see that just by the study of the behavior of zero and infinity we can uncover virtually all critical mathematical constants found in mathematics.

Relating I^{Weak} to I

Our first experiment involves comparing the growth rate of I to I^{Weak}. This one can be solved using intuition with the combination of the *guess and test* methodology. Since I^{Weak} clearly grows to infinity at a much slower rate than I, we must equalize it somehow. We could either raise it to some power, or use it as the power raised to another base. Using the latter, we get very close matching growth rates with I, minus some small value. After doing some quick research, you will see that the missing piece of the puzzle was the Euler-Mascheroni Constant.

```
# DISCOVERY
# 1 + 1/2 + 1/3 + 1/4 + 1/5 + ... / n  = Infinity-Weak (I-W)
# 1 + 1 + 1 + 1 + 1 + 1 + 1 + ...      = Infinity (I)
# e ^ I-W = I*e^Euler–Mascheroni

x=0
iterations = 10000

for i in range(1,iterations):
   x = x + 1/i
print "e^I-W=",numerical_approx((e^x),prec=100)
print "I*e^Euler–Mascheroni=",numerical_approx((iterations-1)*e^0.577215664901532,prec=100)
```

After 10000 Iterations:
e^I-W= 17809.833651114490892196461565
I*e^Euler–Mascheroni= 17808.943107483974017668515444

Thus the relationship becomes:

$$I^{Weak} = \gamma + \ln I$$

Mathematicians have actually studied this relationship in great detail. It is interesting that you can discover the Euler Mascheroni Constant, known as Gamma, yourself by doing this simple experiment above.

Official Formula for Euler Mascheroni Constant:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right) = \int_1^\infty \left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)\,dx.$$

Relating $I^{\mathbf{Weak}}$ to I^{Strong}

Using the same approach as before, we can compare I^{Weak} to the sum of all natural numbers labeled I^{Strong}.

```
#DISCOVERY #2
# 1 + 1/2 + 1/3 + 1/4 + 1/5 + ... / n  = Infinity-Weak (I-W)
# 1 + 2 + 3 + 4 + 5 + 6 + 7 + ...      = Infinity-Strong (I-S)
# e ^ I-W = 2.51882336*sqrt(I-S)
# sqrt(2) * e ^ gamma  = 2.5188

x=0
j=0
iterations = 50000

for i in range(1,iterations):
   x = x + 1/i
   j = j + i
print "e^I-Weak=",numerical_approx((e^x),prec=100)
print "2.5188… X I-Strong=",numerical_approx(2.51882336*sqrt(j),prec=100)
```

Output after 50k iterations:

e^I-Weak= 89052.730364785146020151462583
2.5188… X I-Strong= 89052.963380362243596604071176

Put in terms of gamma, I've discovered the relationship below:

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln\sqrt{\sum_{k=1}^{n}k}\right] - \ln\sqrt{2}$$

To date, I have not seen this formula proven explicitly but it does appear to be valid.

In terms of I^{Weak}:

$$I^{Weak} = \gamma + \ln\sqrt{2} + \ln\sqrt{I^{Strong}}$$

Relating I^{Weak} to $I^{Mult.}$

```
#DISCOVERY 3
# 1 + 1/2 + 1/3 + 1/4 + 1/5 + ... / n  = Infinity-Weak (I-W)
# 2 * 2 * 2 * 2 * 2 * 2 * 2 ....       = Infinity-Strong 2 (I-S2)
# I-W = ln(2.51882336 ln(I-S2))
x=0
j=1
iterations = 10000

for i in range(1,iterations):
    x = x + 1/i
    j = 2 * j
print "I-W=",numerical_approx(x,prec=100)
print "I-S2=",numerical_approx(ln(2.51882336*ln(j)),prec=100)
```

After 10k iterations:

I-W= 9.7875060360443822641784779049
I-S2= 9.7675193182412129281339886819

So the formula becomes:

$$I^{Weak} = \ln(\sqrt{2}e^{\gamma}\ln I^{Mult.})$$

All three of the previous formulas employ the Euler–Mascheroni constant, also known as gamma.

Relating I^{Weak} to $I^{Factorial}$

It is also possible to relate I^{Weak} to the *infinity* based on the Factorial function. After some trial and error, it does appear that the best fit requires nested logarithms of base pi. It is extremely difficult to programmatically verify if there are extra terms missing so I used the approximate notation, although big O notation may be more appropriate.

```
# Discovery 4
# 1 + 1/2 + 1/3 + 1/4 + 1/5 + ... / n  = Infinity-Weak (I-W)
# 1 * 2 * 3 * 4 * 5 * 6 * 7 ....       = Infinity-Factorial (I-F)
# I-W = log(log(I-F,pi),pi)

x=0
j=1
```

```
iterations = 10000

for i in range(1,iterations):
   x = x + 1/i
   j = j * i
print "I-W=",numerical_approx(x,prec=100)
print "I-F=",numerical_approx(log(log(j,pi),pi),prec=100)
```

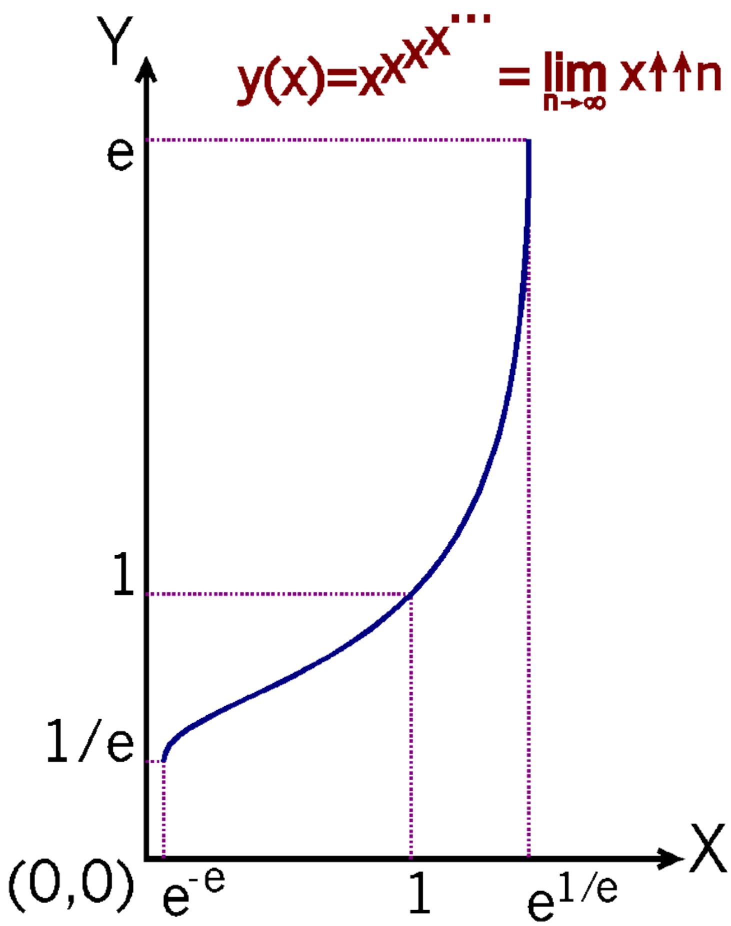
Output after 10k iterations:
I-W= 9.7875060360443822641784779049
I-F= 9.7669513926850843394211228918

The results appear to be on the order of log base pi but further computation is necessary to verify. The most important point is that it is definitely possible to relate each of our concepts of infinity with one another. Below is a back of the envelope approximation.

$$I^{Weak} \approx O(\log_{\pi} \log_{\pi} I^{Factorial})$$

Relating I^{Weak} to $I^{Tetration}$

Tetrations are one of the most computationally-intensive areas of research in mathematics. This flavor of *infinity* is so powerful that it can only be graphed for a handful of values.



Source: http://en.wikipedia.org/wiki/File:Infinite_power_tower.png

Tetration Behavior:
1/e is a very important number in infinite tetrations as well. For example: $x^{x^{\cdot^{\cdot^{x}}}} = e$ when $x = e^{1/e}$.

More interestingly, when $x > e^{1/e}$

Then... $x^{x^{\cdot^{\cdot^{x}}}} \to \infty$

Can you come up with a relationship for how tetrations grow related to I^{Weak}, the Riemann Zeta Function?

In Summary

It is indeed possible to look at various concepts of infinity and make meaningful comparisons. There is no reason to give up on a problem just because the results increase monotonically.

This table summarizes the various identities of *infinity*.

Form of Infinity	**Comparative Rate to I^{Weak}**
$\sum_1^\infty \frac{1}{x} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = I^{Weak}$	$I^{Weak} = I^{Weak}$
$\sum_1^\infty 1 = 1 + 1 + 1 + 1 + 1 + ... = I$	$I^{Weak} = \gamma + \ln I$
$\sum_1^\infty x = 1 + 2 + 3 + 4 + ... = I^{Strong}$	$I^{Weak} = \gamma + \ln \sqrt{2} + \ln \sqrt{I^{Strong}}$
$\prod_1^\infty 2 = 2 * 2 * 2 * 2 * 2 * \dots = I^{Mult.}$	$I^{Weak} \approx \ln(\sqrt{2}e^{\gamma} \ln I^{Mult.})$
$\prod_1^\infty x = 1 * 2 * 3 * 4 * 5 * \dots = I^{Factorial}$	$I^{Weak} \approx O(\log_\pi \log_\pi I^{Fibonacci})$
$2^{2^{2^{\cdots}}}$ Evaluated top to bottom (called a ***tetration***)	Not Discussed

Chapter 5: A New Mathematics System

Introduction

Operations such as division by zero and manipulations of infinity have been barred from mathematics for a simple, but important reason. Currently, there does not exist a system where division by zero does not create a contradiction.

Put differently, our current rules will break down if we allow division by zero (and change nothing else).

But, let's not be discouraged. We will now begin the process of creating a contradiction-free mathematical system. This will be the beginning of a new mathematical system that **does allow division by zero**. The mathematical community will need to extend it further to create a truly consistent contradiction-free formalized system.

This bold new prospect does come at some expense. We will see that certain properties that were thought to be well understood (e.g. the distributive property), must undergo revision to allow this new operation. We will need to bend or change our previous rules in order to create a new mathematical system that has no exceptions (such as disallowing division by zero).

We must follow one very important rule:

Everything must remain logical and without contradiction.

Like the modernization of a home, we must make sure not to damage the good as we do maintenance on the outdated. We hope to disturb as few functioning properties as possible and introduce only the tiniest amount of new concepts necessary to create a more functional system for performing operations.

The following will show how one could design a system of mathematics that is consistent with logic and current mathematical operations. This is intended to be a proof of concept rather than a formalized language.

Redefining Zero and Infinity as a Process

Our first action will be to stop our calculators from returning "Undefined". We are going to need to make some choices. Our first, but critical choice will be to decide what our definition of infinity is.

From our previous chapter, we looked at many series' that increased forever. Let's make our first hard decision and *define infinity* to be the linear version of *infinity*, denoted I.

Let I $(x)= \sum_{k=x}^{\infty} 1$ x+1+1+1+1+1+…

*when x is a positive real number

Let I $(x)= \sum_{k=x}^{\infty} -1$ x-1-1-1-1-1-1-…

*when x is a negative real number

We will go into detail shortly why this summation is a different color. In essence, we start with the lower value, x, and constantly increment the value within the summation. It is very similar to the classic summation operation; we are just beginning with an initial value.

Let 0(x) = $x/\sum_{k=1}^{\infty} 1$ x/(1+1+1+1+1+...)
*when x is a positive or negative real number

In words, this operation can move x closer to origin at the same rate a summation can add 1. We will define it as the inverse of I. We will go into more detail concerning the imaginary numbers.

Furthermore, you will see that infinity and zero behave similarly to the wave function. Once collapsed, information is lost from the original configuration. However, before collapsing the equation, many operations can be performed.

0(x) → 0 The magnitude of 0 is 0.
X must be some number.

Both 0 and I should be viewed as a "process" or "operator" rather than a number. Additionally, we must recognize that mathematics is composed of operations. If you think about it, we are very illogical about how we perform an operation in math today.

We believe that a mathematical operation can be performed infinitely fast in a theoretical world. However, we also believe that there are certain rules that must be respected in terms of order (i.e. the order of operations). These steps must be done to completion before the rest of the problem can be solved. This of itself is not logical. We just stated that a math problem can be solved by performing a finite number of steps in zero time. We should recognize that:

Even in a theoretical world, there must exist some non-zero amount of time, required to perform a single operation.

We introduce time logically by the concept of order. This is an extension of traditional mathematics. Since traditional mathematics requires us to perform certain operations before others during evaluations, it can be argued that it would be illogical for all operations to occur in zero time (completely in parallel). This allows us to introduce a very simple but important concept which allows us to compute the intersection of zero and infinity if they are viewed as processes which take time rather than numbers or points on the number line.

Time for an Example

We will do a simple example of how we can interpret infinity and zero. When *infinity* is used as an operator, it behaves as follows.

$$I(3)=\sum_{k=3}^{\infty} 1 = 3,4,5,6,\ldots$$
$$I(-3)=\sum_{k=3}^{\infty} 1 = -3,-4,-5,-6,\ldots$$

We have created a new operator for ease of expression. It behaves almost exactly as the summation operator, but it begins at some initial value. In this case, k=3 as the initial value. We begin at 3 and then increase at a uniform pace in the positive direction.

However, when x=-3, we continue in the opposite direction.

Let's look at an example using *zero:*

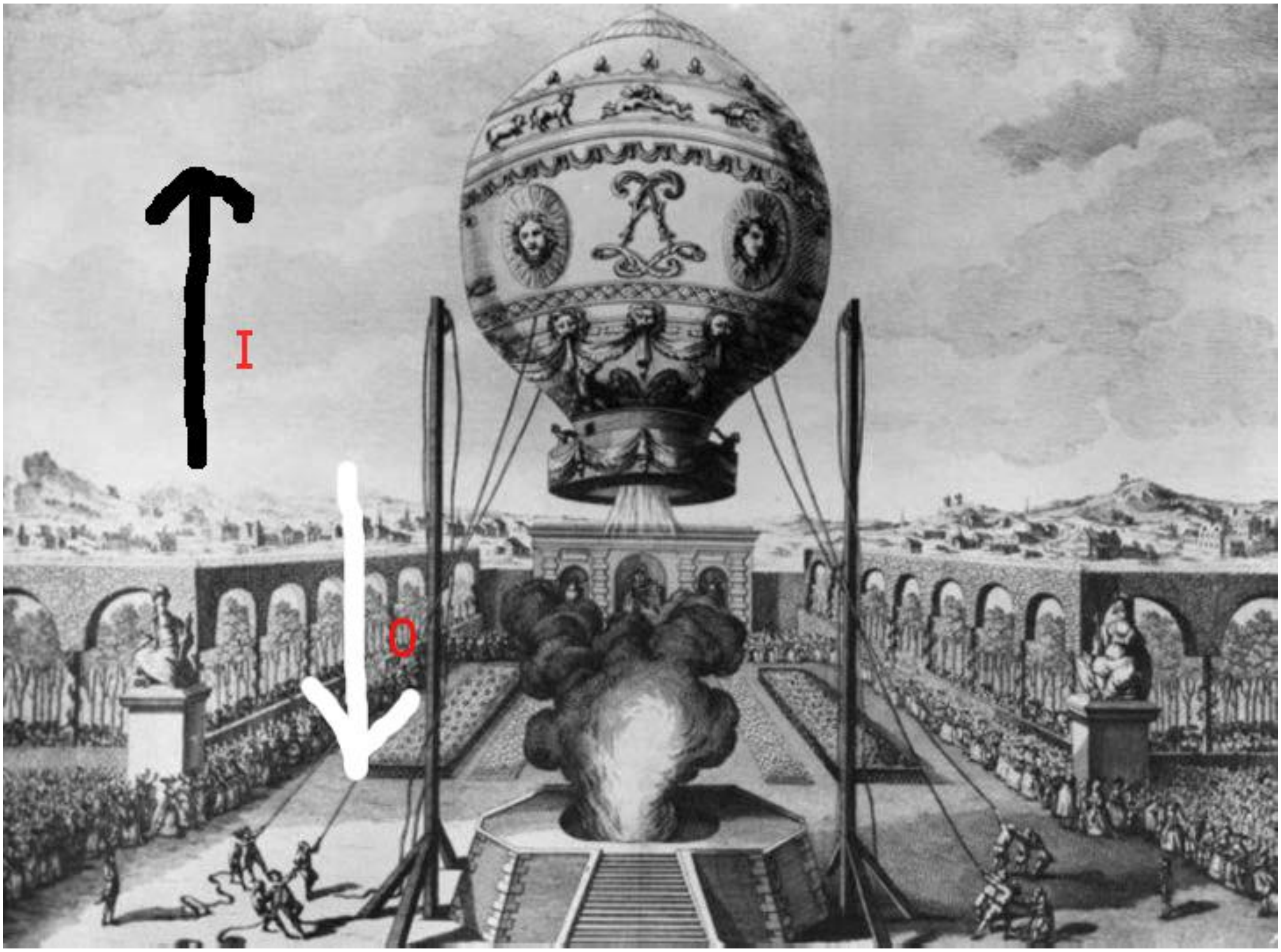
$$0(3)=x/\sum_{k=1}^{\infty} 1 = 3/(1+1+1+1+1+\ldots)$$
$$0(-3)=x/\sum_{k=1}^{\infty} 1 = -3/(1+1+1+1+1+\ldots)$$

In words, we begin with 3 and then divide it at a uniform pace. It is important to note that it takes an infinite amount of time to reach the concept of zero just as it takes infinite amount of operations to reach the concept of infinity.

$$0(\mathrm{I}(3))=0 \bullet \sum_{k=3}^{\infty} 1 \rightarrow \frac{3+1+1+1+1+\cdots}{1+1+1+1+1+\cdots} \rightarrow \frac{\text{Ɨ}(3)}{\text{Ɨ}(1)} \rightarrow 3$$

* We will later address how the distributive property is revised.

In this example, the infinity and zero operators are in constant opposition, per definition. In a real world example, it is like a hot-air balloon that is at 3 feet. The pilot turns on the hot air, which exactly counteracts gravity. The idea of *infinity and zero* is similar to the force of gravity versus a propulsion force. We "define" them to be exactly balanced for our mathematical system, such that infinity is no stronger than zero. This is the necessary step to allow us to compute such products as *infinity times zero*. If we chose *infinity* to be stronger than zero, then the answer could be infinity or some other constant. If we chose *zero* to be stronger than *infinity* we could end up with rules such that anything (including *infinity*) times zero equals *zero*.



It appears that for a mathematical system to be useful, *zero* and *infinity* must be inverses.

θ - The Identity Operator

Interestingly, with zero now being an operator there is now a gap where the original definition of zero once resided. This operator does not define the idea of "nothingness". The Zero Operator defines a destructive force that eventually reduces any number to "nothingness".

Mathematical Symbol	**Operator or Point?**
+ and -, etc	Operator
1,2,3,4,…	Points
-1,-2,-3,-4,…	Points
I	Operator
0	Operator
θ	Operator **and** Point

We have introduced a new operator, θ. In most cases, it attempts to remove itself from the equation. In essence, it adjusts its value to "disappear" from the equation. In some cases, it must equal 1 and in some cases it must equal 0.

$\theta \bullet 1 = 1$	$\theta + 1 = 1$	$1 / \theta = 1$	$\theta! = \theta$	$\theta \bullet I = I$
$\theta^{I(1)} = (0(1) \bullet \ln(I(1)))^{I(1)}$	$e^{\theta} = 1$	$\theta / 1 = 1$	$\theta = \theta$	$\theta \bullet 0 = 0$

It is important to note that the identity operator, θ, must be self-aware and aware of the function around it. In other words, it must "know" about how it is being used with the equation. It must know all this information in order to leave the remaining part of the equation untouched.

Unfortunately, **this** is the operator that mathematicians truly despise. It's not a consistent number. It breaks axioms, the core of our modern mathematics. It breaks the beauty of math! It is all these things and more, but it may be the **necessary** ingredient missing from our system.

Inner Workings of The Identity Operator

How does the *Identity Operator* work? Is there any sound math to support such a concept?

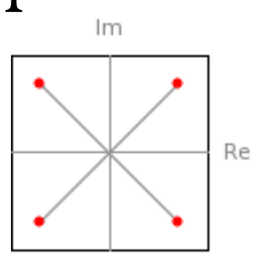
Indeed, it is possible to construct such an operator via an *imaginary unit circle*. You will see that during addition, all values cancel out when added together. During multiplication, you will see that all values can equal one.

We will break down the math for multiplication. You will see that no matter how many points we add, we still are balanced during addition and equal one during multiplication.

4 Points of an Imaginary Unit Circle

$$\sum All\ points = 0$$

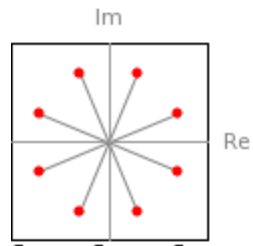
$$\prod All\ points = 1$$



$$\left(\frac{\sqrt{2}}{2} + \sqrt{2} \times \frac{i}{2}\right)\left(-\frac{\sqrt{2}}{2} - \sqrt{2} \times \frac{i}{2}\right)\left(\frac{+\sqrt{2}}{2} - \sqrt{2} \times \frac{i}{2}\right)\left(-\frac{\sqrt{2}}{2} + \sqrt{2} \times \frac{i}{2}\right)$$

Equals 1

8 Points of an Imaginary Unit Circle

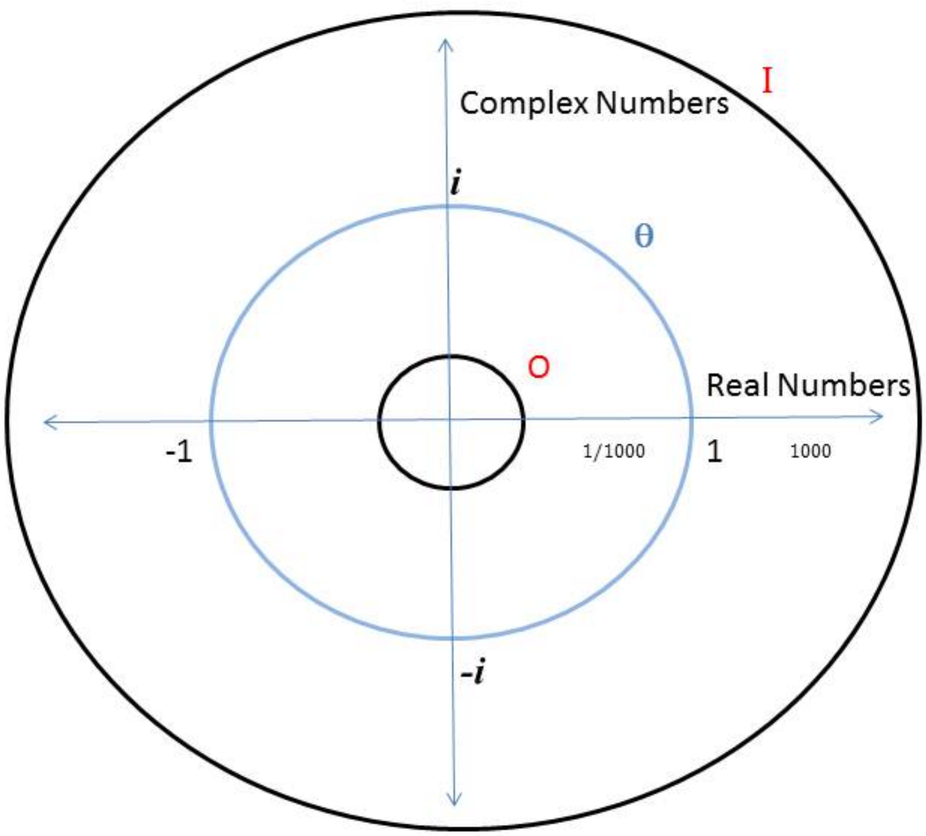


$$e^{1i\times\frac{\pi}{8}}\ e^{3i\times\frac{\pi}{8}}\ e^{5i\times\frac{\pi}{8}}\ e^{7i\times\frac{\pi}{8}}\ e^{9i\times\frac{\pi}{8}}\ e^{11i\times\frac{\pi}{8}}\ e^{13i\times\frac{\pi}{8}}\ e^{15i\times\frac{\pi}{8}}$$

Also, Equals 1

This operator can be seen with the infinity and zero operators as sitting directly in between both "forces" forming a circle in the complex plane.

Visualization of the Identity Operator in the Complex Plane



Properties of I and 0

Our hypothetical operators can be described by example in the table below. Some notable properties are how the infinity and zero operators behave when distributed. Furthermore, addition and multiplication are virtually identical for the infinity and zero operators. We will go into more detail to explain how these operators could behave to keep their behavior consistent across all operations.

I	0
I • 1=I(1)=1+1+1+1+....	0 • 1 = 0
I•-1=I(-1)= -1–1–1–....	0 • -1 = 0
I(1)–I(1)=(1+1+1...) – (1+1+1...) = 0	0(1)–0(1)=0
I(1)+I(1)= I^2 $(1+1+1+1+1+...)^2$	0 + 0 = 0^2
I • I = I^2	0 • 0 = 0^2
1^I = 1	1^0 = 1
I / I = θ	0 / 0 = θ
I^I = 0	0^0 = 1
Properties relating both 0 and I	
0 • I = θ	
0 • -I = θ	
0^I = 0	

$I^0 = 1$
$I(0(1)+0(1)) = \cancel{I(}\cancel{0}(1))+0(1)=\theta(1)+0(1)=1$ See Section: New Distributive Rule – Basic Example
$(1 + 0(1))^I = e^1$ See Section: Exponents – A Deeper Look
$(1 - 0(1))^I = e^{-1}$
$(1 + 0(\theta))^I = e^0$
$(1 - 0(\theta))^I = e^0$
$I^I \bullet 0^0 \bullet 0^I \bullet I^0 = \theta$

New Distributive Rule – Basic Example

Many proofs contradicting division by zero rely on the distributive property. Indeed, we can easily break this mathematical system if we assume the distributive property can be applied to these operators as it exists. However, logically, we need to recognize that the zero and infinity operator behave differently when being distributed through an equation.

$$0 \bullet (I(1)+7) \neq 0 \bullet I + 0 \bullet 7$$

Rather, it fails to distribute itself like a constant and immediately cancels with its counterpart upon multiplication.

$$0 \bullet (I(1) + 7) = \theta(1) + 7 = 8$$

Exponents – A deeper look

Let's look at a practical example of how e shows up in many of our results above. You may already be familiar with the formula for Euler's constant, e.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

We **cannot** solve this problem by evaluating the parenthesis first, and then evaluating the exponent. If we did, we would evaluate the above as 1 to the power of infinity, and thus 1. We must evaluate all in parallel.

We can rewrite the above with our own notation. Using a more explicit form of *Zero* and *Infinity* we can rewrite the above equation:

$$(1 + 0(1))^{I(1)} = e$$

We can obtain e^2 with the following:

$$(1 + 0(2))^{I(1)} = e^2$$

This behavior can continue sequentially:

$$(1 + 0(3))^{I(1)} = e^3$$

but…

$$(1 + 0(I(1)))^{I(1)} \neq e^I$$

Note: we use the I(1) operator because we started our sequence at 1 and increment the series.

$$(1 + \theta \bullet 1)^I = 2^I$$

You will see that limits don't behave in the traditional sense when we deal with infinity.

The pattern that follows with 1,2,3,4,5,6 does not yield the same result when *infinity* is "plugged in" as seen in the table below. To conserve space 0 and I actually represent 0(1) and I(1).

$$\lim_{n\to\infty}\left(1+\frac{n}{n}\right)^n = 2^n$$

$(1 + 0)^I$	$(1 + 2\bullet 0)^I$	$(1 + 3\bullet 0)^I$	$(1 + 4\bullet 0)^I$	$(1 + 5\bullet 0)^I$	$(1 + I(1)\bullet 0)^I$
e	e^2	e^3	e^4	e^5	2^I

The problem we are dealing with is our method on how we traditionally approach infinity. The **incorrect** yet traditional method to approach infinity is as follows:

Traditional way to approach infinity: 1,2,3,4,5,…I(1)

But, if we change the way we look at limits, we can make them work, if we approach them as follows: 1,2,3,4,5,…,$I^{1/I}$, $I^{1/I-1}$, …,$I^{1/3}$, $I^{1/2}$, $I^{2/3}$, $I^{3/4}$, $I^{4/5}$,…I

$(1 + I^{1/2}\bullet 0)^I$	$(1 + I^{2/3}\bullet 0)^I$	$(1 + I^{3/4}\bullet 0)^I$	$(1 + I^{4/5}\bullet 0)^I$	$(1 + I^{5/6}\bullet 0)^I$	$(1 + I\bullet 0)^I$
≈2.65^ $I^{1/2}$	≈2.56^ $I^{2/3}$	≈2.47^ $I^{3/4}$	≈2.40^ $I^{4/5}$	≈2.35^ $I^{5/6}$	2^I *

This table was generated with I=500 to support quicker calculations.

* **not** e^I

Regarding subtraction, we will refer to the classic definition of the inverse of Euler's constant.

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = 1/e$$

Rewriting it using our own notation yields the following equation:

$$(\ 1 + 0(-1)\ )^{I(1)} = 1/e$$

We can see that these values can be approximated by substitution:

$$\left(1-\frac{1}{500\,000}\right)^{500\,000}$$

It continues as the previous equations,

$$(\ 1 + 0(-2)\ )^{I(1)} = 1/e^{2}$$

$$(\ 1 + 0(-3)\ )^{I(1)} = 1/e^{3}$$

Following the right-hand side of the equation we would expect to continue the same pattern. However,

$$(\ 1 + 0(I(-1))\ )^{I(1)} \neq 1/e^{I}$$

Note: we use the I(-1) operator because we started our sequence at -1 and decremented the series. We can evaluate the product of 0 and I to θ. This will evaluate to 1 and (1-θ) will evaluate to θ.

$$(\ 1 + \theta(-1)\ )^{I(1)} = (\theta(0))^{I(1)} = (0(1)\bullet \ln(I(1)))^{I(1)}$$

We can see this empirically if we examine the limits as before approaching them as follows: 1,2,3,4,5,…,$I^{1/I}$, $I^{1/I-1}$, …,$I^{1/3}$, $I^{1/2}$, $I^{2/3}$, $I^{3/4}$, $I^{4/5}$,…I

$(1 - I^{1/2}\bullet 0)^{I}$	$(1 - I^{2/3}\bullet 0)^{I}$	$(1 - I^{3/4}\bullet 0)^{I}$	$(1 - I^{4/5}\bullet 0)^{I}$	$(1 - I^{99/100}\bullet 0)^{I}$
≈(1/2.785)^ $I^{1/2}$	≈(1/2.912)^ $I^{2/3}$	≈(1/3.07)^ $I^{3/4}$	≈(1/3.257)^ $I^{4/5}$	≈(1/19.87)^ $I^{99/100}$

This table was generated with I=500 to support quicker calculations.

$(1 - I^{999/1000}\bullet 0)^{I}$	$(1 - I^{9999/10000}\bullet 0)^{I}$	$(1 - I^{99999/100000}\bullet 0)^{I}$	$(1 - I^{999999/1M}\bullet 0)^{I}$	$(1 - I\bullet 0)^{I}$
≈(1/166) ^ $I^{999/1000}$	≈(1/1617) ^ $I^{9999/10000}$	≈(1/16101)^ $I^{99999/100000}$	≈(1/160923)^ $I^{999999/1M}$	$\approx(\frac{\ln(I)}{I})^{I}$ See below
I=500	I=500	I=500	I=500	I=500

As we approach infinity we must use larger and larger values for I to correctly identify the trend.

$(1 - I^{999/1000}\bullet 0)^{I}$	$(1 - I^{9999/10000}\bullet 0)^{I}$	$(1 - I^{99999/100000}\bullet 0)^{I}$	$(1 - I^{999999/1M}\bullet 0)^{I}$	$(1 - I\bullet 0)^{I}$
≈(1/150.36)^ $I^{999/1000}$	≈(1/1093.2)^ $I^{9999/10000}$	≈(1/8695.46)^ $I^{99999/100000}$	≈(1/72382.91)^ $I^{999999/1M}$	$\approx(\frac{\ln(I)}{I})^{I}$

I=1000	I=10000	I=100000	I=1000000	I=inf.

A traditional math teacher may tell you that zero to any power is still zero. However, this solution tells the rate at which zero is being approached since zero has a more precise definition.

Simple Test of the New Model

If

$$\lim_{n\to\infty}\left(1-\frac{n}{n}\right)^{n} = \left(\theta(0)\right)^{I(1)}$$

$$= \left(0(1)\bullet\ln(I(1))\right)^{I(1)}$$

Then multiplying the inner product by infinity,

$$\lim_{n\to\infty}\left(n\left(1-\frac{n}{n}\right)\right)^{n}$$

Should equal

$$= \left(I(1)\bullet 0(1)\bullet\ln(I(1))\right)^{I(1)}$$
$$= \left(\theta(1)\bullet\ln(I(1))\right)^{I(1)}$$
$$= \left(\ln(I(1))\right)^{I(1)}$$

Indeed, if we carry out the operation by approaching infinity as before we see,

$$\left(1000\left(1-\frac{1000}{1000}\right)\right)^{1000} =$$
$$\approx 6.88346 \times 10\hat{\ }837$$
$$\approx \ln(1000)^{1000}$$

Symmetry of the Identity operator

In cases where the operator is being applied to the identity operator (which also can represent a point), multiple logical solutions exist and the final solution is the intersection of all.

Two examples that were described above can produce various solutions because the identity operator is agnostic to sign (positive, negative,).

$(1 + 0(\theta))^{I} = e^{-1}, e^{1}, e^{-i}, e^{i}\ldots = e^{\theta} = 1$
$(1 - 0(\theta))^{I} = e^{-1}, e^{1}, e^{-i}, e^{i}\ldots = e^{\theta} = 1$

In this example, since *Zero* is not operating on any value, it is assumed to operate on θ. You will also see the *Zero* and *Infinity* operator are symmetrical on all axis'. Thus all the following operations are equivalent.

$$0(\theta)=-0(\theta)=i\bullet 0(\theta)=-i\bullet 0(\theta),\ldots$$
$$I(\theta)=-I(\theta)=i\bullet I(\theta)=-i\bullet I(\theta),\ldots$$

More on Exponents

We will now study the more general case of *Zero* and *Infinity*. This is the most unbiased form of the operator, the form operating on the identity operator only.

$$0(\theta) \text{ and } I(\theta)$$

In particular, we will look at *zero* to the power of *infinity*. We will notice that the exponential growth rate has an impact on the overall solution. We will approach the solution using the different values of the *identity operator*. Namely, *Zero* can, for the sake of representation, be constructed as follows.

$$0(\theta) = \lim_{n\to\infty}\frac{1}{n}, \lim_{n\to\infty}\frac{-1}{n}, \lim_{n\to\infty}\frac{i}{n}, \lim_{n\to\infty}\frac{-i}{n}, \ldots$$
$$I(\theta) = \lim_{n\to\infty} n, \lim_{n\to\infty} -n, \lim_{n\to\infty} i*n, \lim_{n\to\infty} -i*n, \ldots$$

This can be extended to all infinite points on an imaginary circle. However, in our studies we only need a small set (4 or more) to get a general picture of the behavior of such a function.

As we evaluate the concept of 0^I, we keep in mind that our result is the superposition of all possible definitions used for *Zero* and *Infinity.*

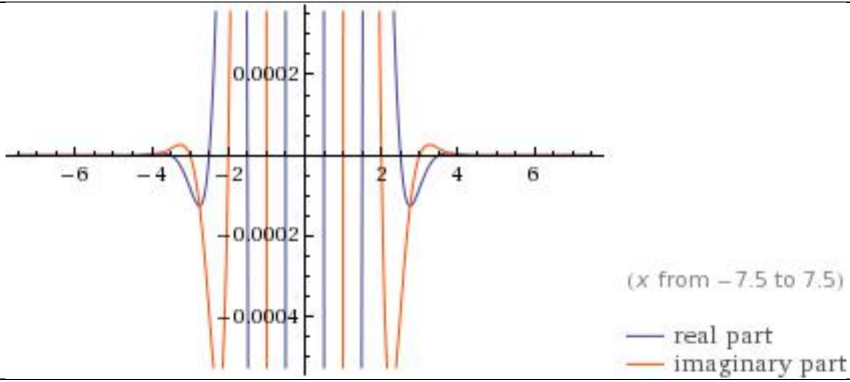
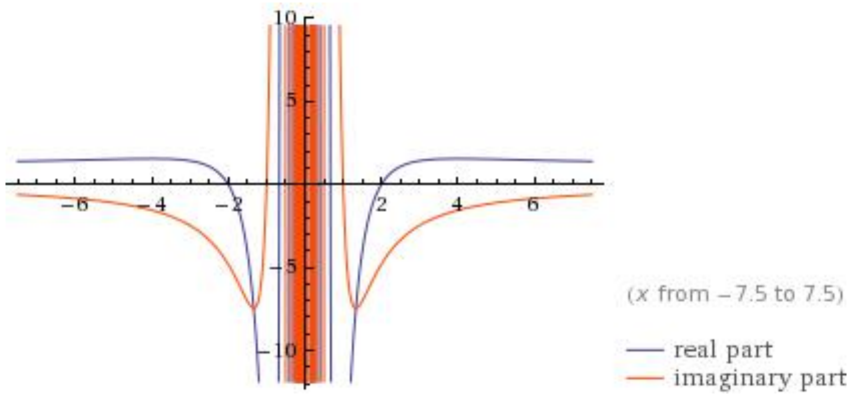
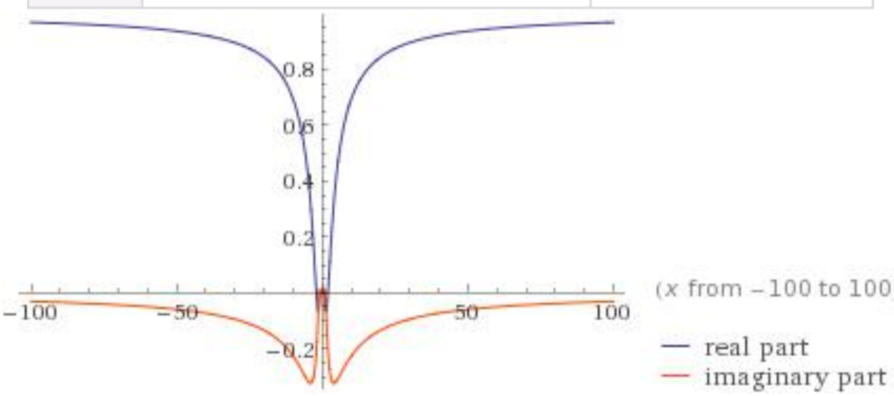
Due to computational resources, we will only use 4 of the possible definitions for zero (the first four listed above). We will do the same for infinity. Thus, our equation for becomes:

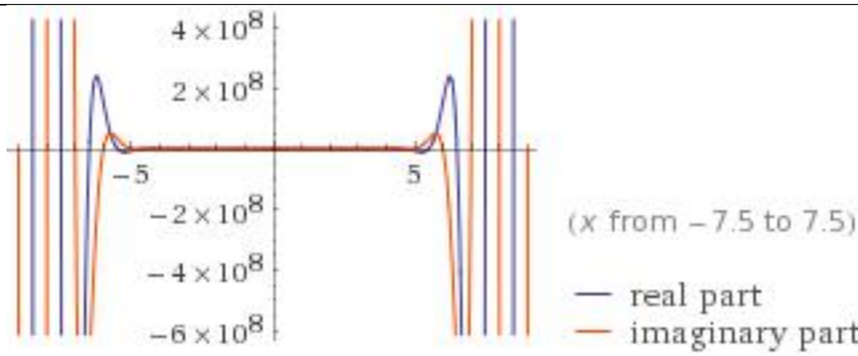
$$\left(\frac{1}{x}\right)^{x}\left(\frac{1}{i\,x}\right)^{x\,i}\left(-\frac{1}{x}\right)^{-x}\left(-\frac{1}{i\,x}\right)^{-i\,x}$$

As x approaches traditional positive infinity (∞), our $I(1)$.

But, before we take a look at a large value for x, let's examine the values near origin, just for the sake of curiosity.

Function	Graph	Limit	Simplified
I^I	plot $(i\,x)^{i\,x}\,(-i\,x)^{-i\,x}\,(-x)^{-x}\,x^{x}$ $x = -7.5$ to 7.5	0	$e^{(-1-i)\pi x}$

			
I^0	plot $(i\,x)^{\frac{1}{i\,x}}\,(-i\,x)^{-\frac{1}{i\,x}}\,(-x)^{-1/x}\,x^{1/x}$ $x = -7.5$ to 7.5 	1	$e^{(1-i)\pi/x}$
0^0	plot $\left(-\frac{1}{x}\right)^{-1/x}\left(-\frac{i}{x}\right)^{-\frac{i}{x}}\left(\frac{i}{x}\right)^{\frac{i}{x}}\left(\frac{1}{x}\right)^{\frac{1}{x}}$ $x = -100$ to 100 	1	$e^{-(1+i)\pi/x}$
0^I	plot $\left(-\frac{1}{x}\right)^{-x}\left(-\frac{i}{x}\right)^{i\,x}\left(\frac{i}{x}\right)^{-i\,x}\left(\frac{1}{x}\right)^{x}$ $x = -7.5$ to 7.5	I	$e^{(1-i)\pi x}$

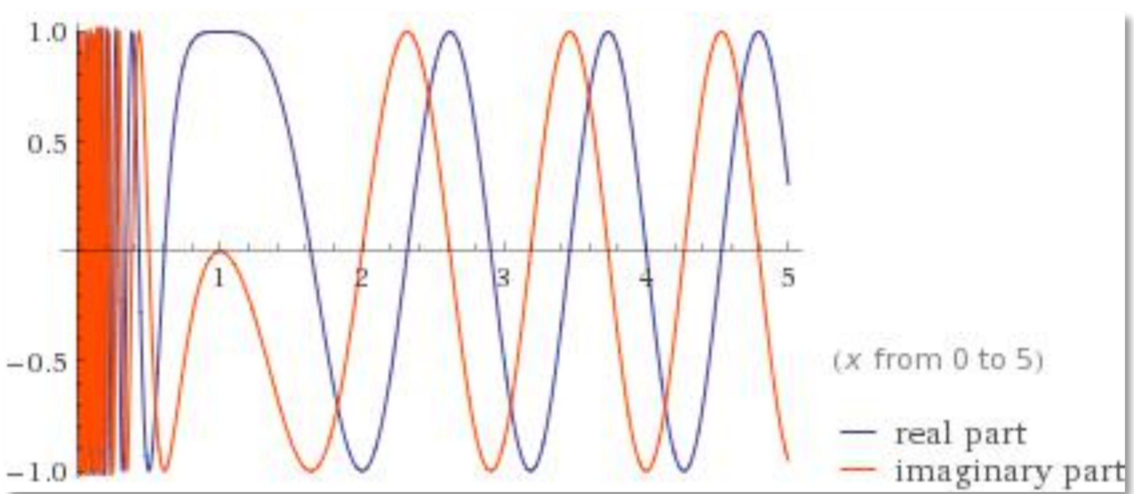
			

From the above, we also learn that:

$$I^{I} \bullet 0^{0} \bullet 0^{I} \bullet I^{0} = \theta$$

Graphically, we can plot the following function:

$$e^{(1-i)\pi x}\, e^{(-1-i)\pi x}\, e^{(1-i)\times\frac{\pi}{x}}\, e^{-(1+i)\times\frac{\pi}{x}}$$



Perhaps, this symmetry was a necessary starting condition for the universe.

Final Example

Let's me as you a simple question. What set of numbers is larger?

$\prod_{n=1}^{\infty} 2n$ = 2 * 4 * 6 * 8 *10 * 12 * ....

OR

$\prod_{n=1}^{\infty} (2n-1)$ = 1 * 3 * 5 * 7 * 9 * 11 * ....

At first glace you could say that the even numbers are larger. When comparing a ratio of the even number to the odd, you can see that the even numbers are always 1 "term" greater than the odd numbers.

$$\frac{2*4*6*8*10*12*....}{1*3*5*7*9*11*....}$$

Isn't 2 > 1, and 4 > 3, and 6 > 5, and so on?

Is the product of even numbers, truly larger than the product of the odds?

Wait...

My intuition tells me, they should be equal; isn't there another way to look at this. Shouldn't there be symmetry in the universe?

When we multiply the odds by 1, it really doesn't do anything. The product of odd numbers really should start with multiplying 3 by the rest of the numbers. Let see what happens when we do that...

The product of all the odd numbers could really be rewritten as shown:

$\prod_{n=1}^{\infty}(2n+1)$ = 3 * 5 * 7 * 9 * 11 * 13 * ....

After all, who needs a silly multiplication by "1" when we are really trying to get to the *heart* of the product. Logically, I could look at the ratio of products of the evens to the odds this way:

$$\frac{2*4*6*8*10*12*....}{3*5*7*9*11*13*....}$$

Wait...

This is confusing as well. Now, the odds appear to be larger than the evens. Hmmm...maybe if you look at it one way the odds are smaller than the evens, and if you look at it another way the odds are larger?

When I showed these results to my brother, he said, it appeared to be a paradox. Should we close the case or can we look further into this?

Indeed, the rabbit hole goes much deeper than you think. Let us explore...

When the odds are grouped starting with "1", the ratio of evens to odds approaches infinity.

When the odds are grouped starting with "3", the ratio of evens to odds approaches 0.

Let us as the following question:

What happens when you multiply this idea of infinity and zero?

$$\frac{2*4*6*8*10*12*....}{1*3*5*7*9*11*....} \times \frac{2*4*6*8*10*12*....}{3*5*7*9*11*13*....} = ?$$

This is a fascinating question. If you believe in symmetry, you would believe that zero doesn't overpower infinity; and likewise, infinity does not overpower zero. So this gives us our first glimpse at perhaps an important mathematical constant.

In 1655, John Wallis discovered the answer to our question. It is famously called the *Wallis Product*. However, John Wallis may not have looked at our problem in the same light.

In mathematics, the Wallis Product states that:

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1} \bullet \frac{2n}{2n+1} = \frac{2\bullet2\bullet4\bullet4\bullet6\bullet6\bullet...}{1\bullet3\bullet3\bullet5\bullet5\bullet7\bullet...} = \pi/2$$

If we incorporate our new concepts of nothingness we can achieve a symmetry previously undiscovered.

$$\frac{1\cdot3\cdot5\cdot7\cdot9\cdot...}{\theta\cdot2\cdot4\cdot6\cdot8\cdot...} \times \frac{1\cdot3\cdot5\cdot7\cdot9\cdot...}{2\cdot4\cdot6\cdot8\cdot10\cdot...} = {}^{2}/_{\pi}$$

Having this new value in the number system can make other equations more logical.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Logically, it seems quite counter-intuitive, but the factorial of 0 must be 1 for the above equation to work. But replacing 0 with θ may make it more intuitive to solve.

Important Takeaway

Indeed, it can be very difficult to create a contradiction-free operator that can be integrated into our mathematics system. However, we have seen that if we are willing to redefine certain rules, it is possible to modify our system to create an equally powerful or even more powerful model to describe numbers.

The key goals of this extension of mathematics is:

- To be able to perform **without exception** addition, multiplication and exponentiation and their respective inverse, at any value.
- To be able to solve problems which have substantial calculations involving infinite series'.
- To identify a self-aware substructure **inherent** to the mathematical system. This is necessary since our universe contains self-aware substructures (i.e. consciousness).
- To be able to identify logical starting conditions of the universe.

The key mechanisms to implementing these goals are:

- To describe infinity and zero as a **process** (i.e. operator) rather than fixed points that reside somewhere on the number line. Behaviors of these operators must be defined and checked for inconsistencies (i.e. contradictions).
- To create a unique point for the idea of "nothingness" that is independent of the Zero operator.
- To observe that this idea of "nothingness" closely resembles the identity operator and that it must be aware of the overall equation to define itself. Therefore, it must be both self-aware and aware of its environment.
- To recognize that an empty universe and a universe of all possibilities are equally logical. If so, this intersection represented by (I and O), could explain our starting conditions.

Caveats

I recognize that this description is in no way a complete definition for all the rules and behaviors of these new concepts. I have, however, noticed that each time I refer back to this problem, I can add more and more consistent rules and mechanisms to truly define behaviors. The system becomes increasingly more useful to answer previously unexplained behaviors. I recognize that more definitions and behaviors must be defined (e.g. logarithm functions) to have a usable system. While some of this work clearly has value, I recognize that some parts are in very early stages and are in need of further revision.

So, do you believe there are Zero possibilities?